# Cellular solid structures with unbounded thermal expansion

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**Abstract**

Material microstructures are presented which can exhibit coefficients of thermal expansion larger than that of either constituent.

Thermal expansion of materials is of considerable practical interest since materials in service may experience a considerable range of operating temperature. Materials with microstructure, including composites, honeycombs, and foams (which are composites with one void phase) are finding increasing application for many purposes. It is therefore of interest to explore the range of thermal expansion coefficient attainable in materials with microstructure.

The constitutive equation for a linear isotropic thermoelastic continuum is [1]:

$$\epsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - [\frac{\nu}{E} \sigma_{ii} - \alpha T] \delta_{ij} , \qquad (1)$$

in which $\epsilon_{ij}$ is strain, $\sigma_{ij}$ is stress, $\nu$ is Poisson's ratio, E is Young's modulus, $\alpha$ is the thermal expansion coefficient, $\delta_{ij}$ is Kronecker's delta, and the usual index notation is used.

Thermal expansion of crystalline solids is considered to arise from the anharmonicity of the interatomic potential. Thermal expansion coefficients for solids are on the order of 3 x $10^{-6}$/°C for ceramics, $10^{-5}$/°C for metals, and $10^{-4}$/°C for polymers [2]; many specific values are given in reference [3]. Thermal expansion of foams was discussed briefly by Gibson and Ashby [2] who pointed out that for a foam viewed as a framework (of homogeneous ribs), the thermal expansion coefficient is the same as the solid from which it is made. An exception is a closed cell foam of very small solid volume fraction, for which thermal expansion is enhanced by the expansion of the gas within the cells [2]. An upper bound for this effect can be estimated as follows. If the foam is very compliant, expansion of the gas occurs at approximately constant pressure. Considering the ideal gas law PV = nRT, the expansion of such a compliant foam at room temperature (T = 300 K) can be no more than $\alpha$ = 1.1 x $10^{-3}$/°C, greater than that of a solid polymer.

For arbitrary two-phase composite materials, bounds have been developed for the thermal expansion coefficient $\alpha$ of the composite in terms of that of each constituent [4,5]; the upper bound is a rule of mixtures. In deriving these bounds it was assumed that there are only two phases, and there is no empty space in the composite. In the present work, it is shown that if one relaxes the

assumption of no empty space, it is possible to generate cellular micro-structures which exhibit a thermal expansion coefficient much larger than that of either constituent.

Consider a cellular solid consisting of bi-layer ribs. The rib element in the cellular solid consists of a laminate of two layers of differing thermal expansion coefficient $\alpha$. The layers have Young's moduli $E_1$ and $E_2$, Poisson's ratio $\nu_1$ and $\nu_2$; thermal expansion coefficient $\alpha_1$ and $\alpha_2$; and thicknesses $h_1$ and $h_2$. The curvature $\kappa$ (the inverse of the radius of curvature $\rho$) of a thin bi-layer elastic beam is given in terms of the temperature change $\Delta T$ by Timoshenko [6], neglecting end effects [7].

The curvature may be written

$$\kappa = 6(\alpha_2 - \alpha_1)\,\Delta T \,\frac{(1 + \frac{h_1}{h_2})^2}{(h_1 + h_2)}\;\frac{1}{3((1 + \frac{h_1}{h_2})^2) + (1 + \frac{h_1}{h_2}\frac{E_1}{E_2})\,((\frac{h_1}{h_2})^2 + \frac{h_2 E_2}{h_1 E_1})}. \tag{2}$$

The curvature can be made arbitrarily large by making the strip thickness h small. The ratio of Young's moduli has little effect on the curvature [6].

We determine the thermal expansion coefficient of cellular solids with two-layer rib elements. Consider first the deformations and strains caused by rib curvature due to strain. The transverse displacement $\delta$ at the midpoint of an initially straight rib of length $l$ is [6]

$$\delta = \frac{\kappa l^2}{8}. \tag{3}$$

Cellular solids which exhibit global thermal expansion due to such transverse rib displacements must have an unusual connectivity, such as that shown in Fig. 1. It is easier to make use of longitudinal displacement of rib elements as determined as follows. Referring to Fig. 2, consider now a curved rib element of arc length $l_{arc}$, radius of curvature r, and included angle $\phi$, in radians. The rib length is given by

$$l = l_{arc}\,\frac{2}{\phi}\,\sin\frac{\phi}{2}. \tag{4}$$

The strain of a unit cell due to end to end length changes in the rib is

$$\varepsilon = \frac{d l}{l}. \tag{5}$$

The effect of a curvature increment $d\phi$ due to bending is

$$\varepsilon = [\frac{1}{2}\,\mathrm{ctn}\frac{\phi}{2} - \frac{1}{\phi}]\,l_{arc}\,d\phi. \tag{6}$$

Eq. 2 gives the curvature of a straight rib, or the increment in curvature of an initially curved rib. Eq. 6 shows that there must be some initial curvature or no finite length change occurs. Cellular solids which expand due to longitudinal displacement by bending of ribs with initial curvature are shown in Fig. 3.

The thermal expansion coefficient of the cellular solid structure is $\alpha = \epsilon / \Delta T$ (at constant stress, see Eq. 1), so

$$\alpha = \frac{l_{arc}}{(h_1 + h_2)} \frac{6(\alpha_2 - \alpha_1)(1 + \frac{h_1}{h_2})^2}{3((1 + \frac{h_1}{h_2})^2) + (1 + \frac{h_1}{h_2}\frac{E_1}{E_2})((\frac{h_1}{h_2})^2 + \frac{h_2 E_2}{h_1 E_1})} [\frac{1}{2}\cot\frac{\phi}{2} - \frac{1}{\phi}] \tag{7}$$

The ratio of rib arc length to rib total thickness can be made arbitrarily large, by making the ribs very slender. The initial curvature of the ribs, as quantified by the included angle $\phi$, should also be made large in order to maximize the magnitude of $\alpha$ given by Eq. 7 (Fig. 4). The thermal expansion coefficient of the cellular solid structure is therefore unbounded. It can be made much larger in magnitude than the thermal expansion coefficient of either solid constituent.

The sign of the thermal expansion coefficient of the cellular solid is governed by the placement of the constituents within each rib. Referring to Fig. 3, if the constituent with the higher thermal expansion coefficient is on its concave side, an increase in temperature will cause the rib to straighten, giving rise to a positive thermal expansion coefficient for the cellular solid. Conversely, if the constituent with the higher thermal expansion coefficient is on its convex side, an increase in temperature will cause the rib to curve more and become shorter, giving rise to a negative thermal expansion coefficient for the cellular solid. The structure shown in Fig. 3 has two dimensional cubic symmetry giving rise to anisotropic elasticity but isotropic thermal expansion. Hexagonal structures which are elastically isotropic in plane are also possible.

The examples presented are two-dimensional honeycombs. They could be produced by a co-extrusion process, by progressive lamination, or by lithography. One may also envisage three-dimensional cellular solids, or foams, containing bi-layer ribs, which may be sintered together. Materials based on polymer bi-layers can deliver higher thermal expansion than those based on metallic bi-layers.

These structures provide a further illustration of the importance of void space in determining physical properties. Inclusion of void space permits one to substantially exceed bounds for thermal expansion developed under the assumption of no void space. Control of the shape of void space has led to materials with a negative Poisson's ratio [8] and lightweight hierarchical materials with high ratios of strength to weight [9]. More developments in materials with unusual or extreme properties are anticipated.
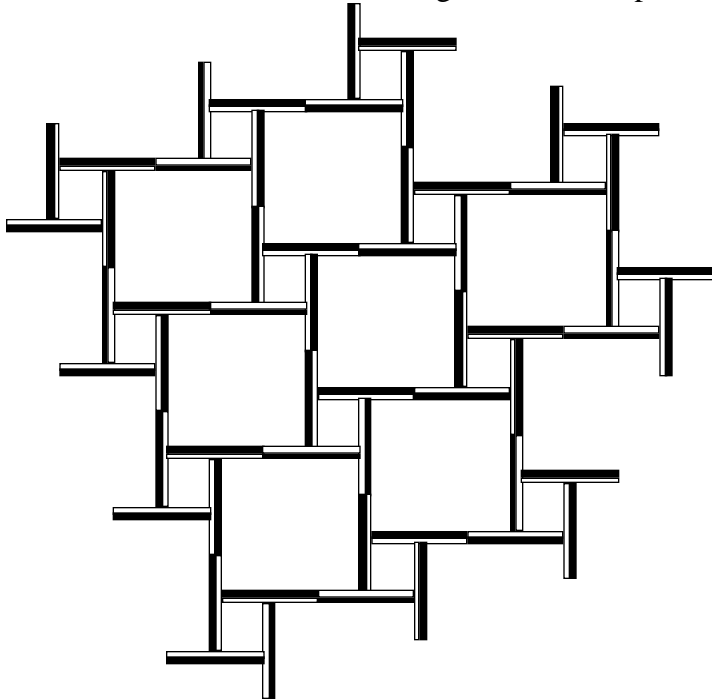
To conclude, arbitrarily high values of thermal expansion can be attained by preparing cellular solids with ribs consisting of a bi-layer containing materials of two different expansion coefficients.
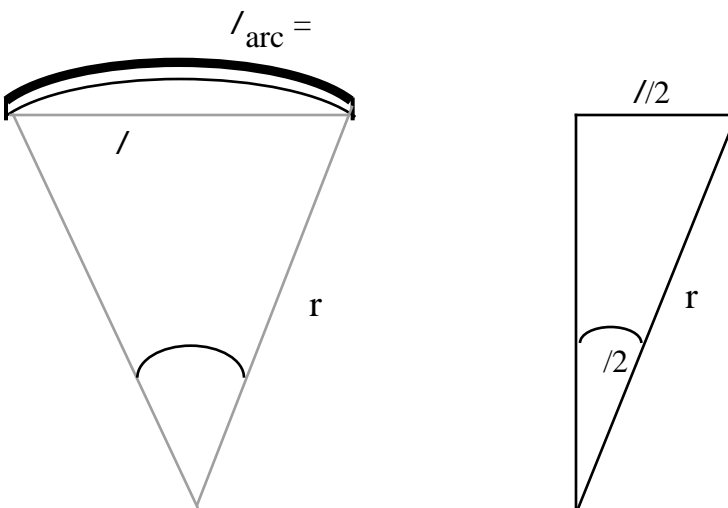
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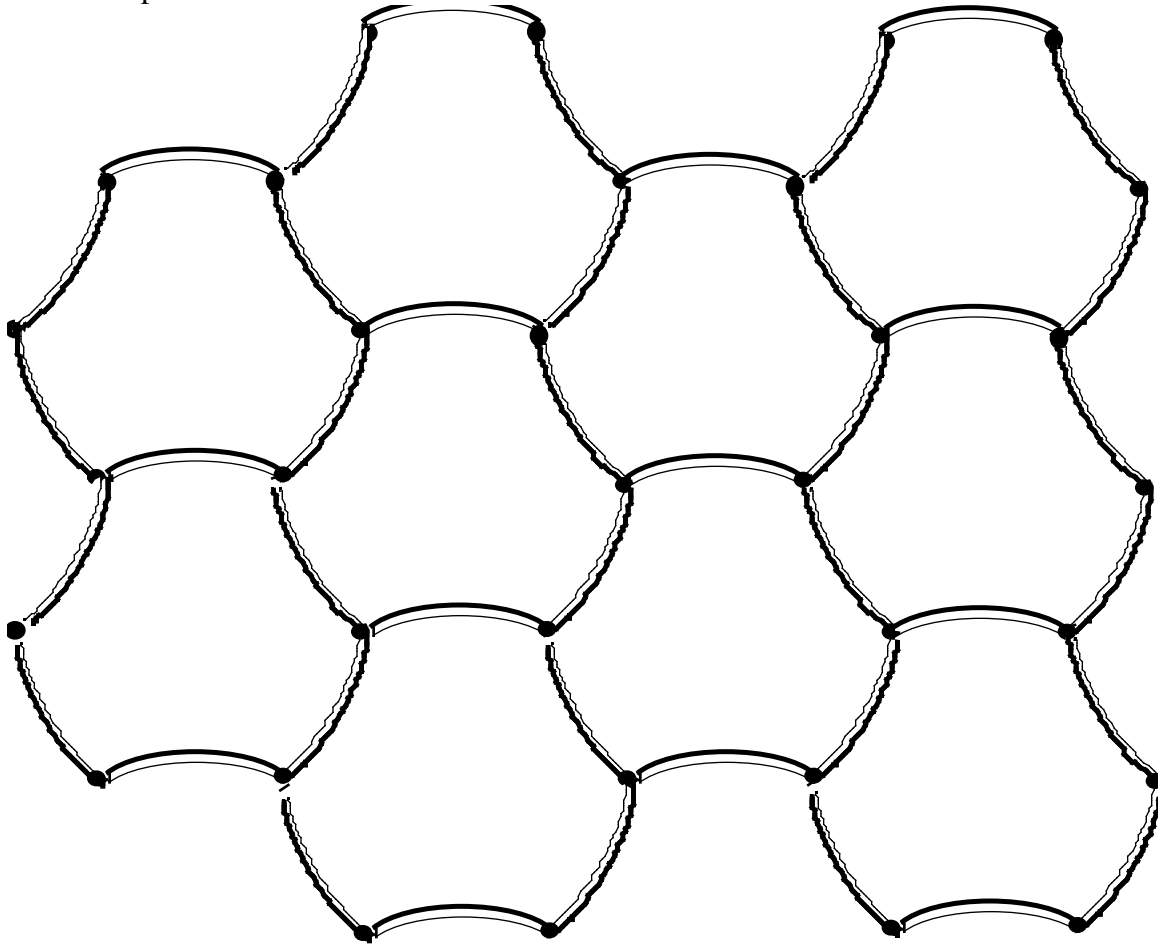
**Figures**

1    Cellular solid which undergoes thermal expansion via lateral bending displacement of ribs.



2    Rib element geometry with initial curvature.

3       Cellular solid with curved ribs. It undergoes thermal expansion via longitudinal bending displacement of ribs.

4        Normalized thermal expansion $\dfrac{}{\alpha_2 - \alpha_1}$ of cellular solid vs. rib curve angle    (in radians)

and rib aspect ratio $\dfrac{l_{arc}}{h}$ for $h_1 = h_2 = h$ and $E_1 = E_2$.