LINEAR ALGEBRA FOR TEST AND ANALYSIS

IMAC - XIX

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MOTIVATION

The use of matrix and vector algebra is an absolute requirement for the efficient manipulation of the large sets of data that are fundamental to applications in structural dynamics, both test and analysis.

Primary problems to be solved:

$$[A]\{x\} = \{b\}$$

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$

LECTURE AGENDA

- Common Nomenclature and Definitions
- Solution of Determined Sets of Equations
- Solution of Overdetermined Sets of Equations
- Solution of UnderDetermined Sets of Equations
- Example Applications

NOMENCLATURE

$[A]$ - matrix

$\{x\}$ - vector

n - number of rows (equations)

m - number of columns (unknowns)

$[A]^T$ - matrix transpose

$[A]^H$ - matrix Hermitian transpose

$[A]^{-1}$ - matrix inverse

$[A]^+$ - matrix generalized inverse

$*$ - complex conjugate

MATRIX EQUATIONS, THREE CASES CAN OCCUR

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1}$$

1. **Underdetermined**: $n < m$	Optimization FEM updating Projection of data onto subspaces
2. **Determined**: $n = m$	Analytical structural dynamics using finite element models
3. **Overdetermined**: $n > m$	Time and frequency domain parameter estimation Least squares applications Sensor placement algorithms

BASIC DEFINITIONS

Matrix: A matrix is an array of numbers. Entries can be referred to by their row and column location.

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

Vector: A vector is a special case of a matrix with either one row or one column.

$$\{b\} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{Bmatrix}$$

Column subscript is dropped

CAN TRANSFORM A SET OF ALGEBRAIC EQUATIONS TO A SINGLE MATRIX EQUATION

$$\begin{aligned}
&3x_1 - 2x_3 = 2 \\
&-2x_1 + 5x_2 - 4x_3 = -1 \\
&7x_1 - 3x_3 = 3 \\
&4x_1 + 6x_2 + 2x_3 = -5
\end{aligned}$$

Equivalent to:

$$\begin{bmatrix} 3 & 0 & -2 \\ -2 & 5 & -4 \\ 7 & 0 & -3 \\ 4 & 6 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \\ 3 \\ -5 \end{Bmatrix}$$

No Exact
Solution

RULES FOR MATRIX OPERATIONS

Multiplication by a scalar:

$$k[A]=\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

Multiplication of a matrix by a matrix:

$$[A][B]\neq[B][A]$$ NOT cummutative in general

$$([A][B])[C]=[A]([B][C])$$ Associative

$$([A]+[B])([C]+[D])=[A][C]+[A][D]+[B][C]+[B][D]$$

Matrix cancellation:

$$[A][B]=[0]$$

Implies one of following

$$[A]=[0]$$

$$[B]=[0]$$

$[A]$ and $[B]$ singular

MATRIX MULTIPLICATION

$$[A][B]=[C]$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{31} & a_{31} & a_{31} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} \\ c_{21} & \boxed{c_{22}} \\ c_{31} & c_{32} \end{bmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42}$$

$$c_{ij} = \sum_k a_{ik}b_{kj}$$

SPECIAL MATRICES

Identity Matrix:

1’s on the diagonal, zeros everywhere else.

$$[I]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Zero Matrix:

Zeros in all locations.

$$[0]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

MATRIX TRANSPOSES

Transpose: Interchange rows and columns.

$$[A]=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \qquad [A]^T=\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix}$$

Hermitian Transpose: Interchange rows and columns and then take complex conjugate of each element.

$$[A]=\begin{bmatrix}1+i & 2 & 3i\\ 4-4i & 5 & 6\\ 7 & 8+8i & 9\end{bmatrix} \qquad [A]^H=\begin{bmatrix}1-i & 4+4i & 7\\ 2 & 5 & 8-8i\\ -3i & 6 & 9\end{bmatrix}$$

SPECIAL MATRIX FORMS

Symmetric: $[A]=[A]^T$

Skew Symmetric: $[A]=-[A]^T$

$$[A]=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

$$[A]=\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -5 \\ -3 & 5 & 0 \end{bmatrix}$$

All diagonal terms 0

Hermitian: $[A]=[A]^H$

Skew Hermitian: $[A]=-[A]^H$

$$[A]=\begin{bmatrix} 1 & 4+3i & 5i \\ 4-3i & 2 & 2+i \\ -5i & 2-i & 0 \end{bmatrix}$$

$$[A]=\begin{bmatrix} -i & 4+4i & -7 \\ -(4-4i) & 5i & -(8-8i) \\ 7 & 8+8i & 0 \end{bmatrix}$$

All diagonal terms real

All diagonal terms imaginary or 0

SPECIAL MATRIX FORMS

Orthogonal $[A][A]^T = [A]^T[A] = [I]$

Unitary $[A][A]^H = [A]^H[A] = [I]$

Idempotent $[A]^m = [A]$ any integer m

Nilpotent $[A]^k = [0]$ for some integer k

SPECIAL MATRIX FORMS

Diagonal:

$$[A]=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$$

Triangular:

$$[A]=\begin{bmatrix}1 & 4 & 5\\ 0 & 2 & 6\\ 0 & 0 & 3\end{bmatrix}$$

Upper
triangular

TOEPLITZ MATRIX

All elements on any superdiagonal and subdiagonal are equal.

$$[T]=\begin{bmatrix} t_1 & t_6 & t_7 & t_8 & t_9 \\ t_2 & t_1 & t_6 & t_7 & t_8 \\ t_3 & t_2 & t_1 & t_6 & t_7 \\ t_4 & t_3 & t_2 & t_1 & t_6 \\ t_5 & t_4 & t_3 & t_2 & t_1 \end{bmatrix}$$

Does not have to be square.

TOEPLITZ MATRIX EXAMPLE

Discrete Time
Invariant SISO System:

$$\{x(k+1)\} = [A]\{x(k)\} + [B]u(k)$$
$$y(k) = [C]\{x(k)\} + Du(k)$$

$\Rightarrow$

Combine to produce
Convolution equation

$$y(k) = \sum_{i=0}^{k} H(k)u(k-i)$$

Can write convolution equation in matrix form.

$$\begin{bmatrix} H_0 & H_1 & H_2 & H_3 & H_4 \\ 0 & H_0 & H_1 & H_2 & H_3 \\ 0 & 0 & H_0 & H_1 & H_2 \\ 0 & 0 & 0 & H_0 & H_1 \\ 0 & 0 & 0 & 0 & H_0 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_3 \\ u_2 \\ u_1 \\ u_0 \end{Bmatrix} = \begin{Bmatrix} y_4 \\ y_3 \\ y_2 \\ y_1 \\ y_0 \end{Bmatrix}$$

For MIMO systems, H_i are matrices, producing a block Toeplitz matrix

Time domain identification techniques

HANKEL MATRIX

All elements on any superdiagonal and subdiagonal perpendicular to the main diagonal are equal.

$$[H]=\begin{bmatrix} h_1 & h_2 & h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 & h_5 & h_6 \\ h_3 & h_4 & h_5 & h_6 & h_7 \\ h_4 & h_5 & h_6 & h_7 & h_8 \\ h_5 & h_6 & h_7 & h_8 & h_9 \end{bmatrix}$$

Does not have to be square.

HANKEL MATRIX EXAMPLE

Modal parameter estimation using Eigensystem Realization Algorithm (ERA)

For SISO system, form Hankel matrix:

h_i is ith Markov parameter
derived from measured FRF
or free-decay data

MIMO results in a block
Hankel matrix

$$[H(0)] = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & \cdots \\ h_2 & h_3 & h_4 & h_5 & \cdots \\ h_3 & h_4 & h_5 & h_6 & \cdots \\ h_4 & h_5 & h_6 & h_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A singular value decomposition (SVD) is performed that then leads to estimates of the modal parameters.

VANDERMONDE MATRIX

First column is 1's with successive columns being the second column with its elements raised to increasing integer powers.

$$[V] = \begin{bmatrix} 1 & v_1 & v_1^2 & v_1^3 \\ 1 & v_2 & v_2^2 & v_2^3 \\ 1 & v_3 & v_3^2 & v_3^3 \\ 1 & v_4 & v_4^2 & v_4^3 \\ 1 & v_5 & v_5^2 & v_5^3 \end{bmatrix}$$

Nonsingular iff
All v_i distinct

Occurs in curve fitting and some frequency domain parameter estimation methods.

VANDERMONDE MATRIX EXAMPLE

Fit a polynomial to a set of data points either test or analytical.

$$\{y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5\}^T$$

Find an expression for *y* as a function of variable *x*

$$y = p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Generate matrix equation using *x*-*y* pairs.

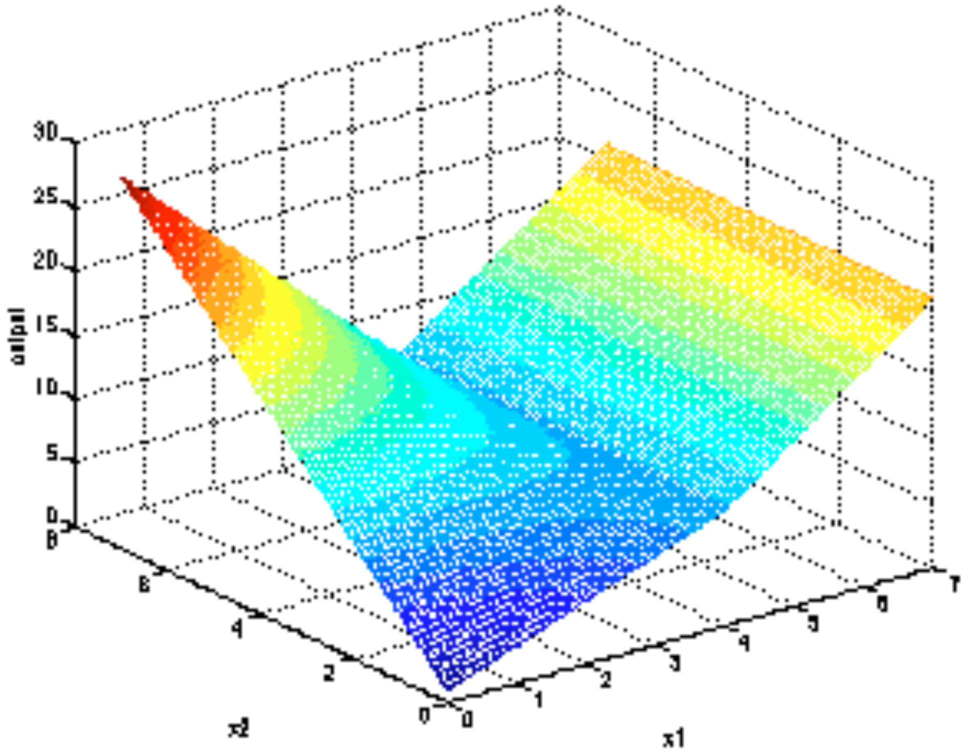
$$y_i = p(x_i)$$

$$\begin{aligned} y_1 &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 \\ y_2 &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3 \\ y_3 &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^3 \\ y_4 &= a_0 + a_1 x_4 + a_2 x_4^2 + a_3 x_4^3 \\ y_5 &= a_0 + a_1 x_5 + a_2 x_5^2 + a_3 x_5^3 \end{aligned}$$

$$\Longrightarrow$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \\ 1 & x_5 & x_5^2 & x_5^3 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix}$$

RESPONSE SURFACES CAN BE SYNTHESIZED TO ACT AS SURROGATES FOR COMPLEX NUMERICAL SIMULATIONS



Super-Metamodel synthesized from two models with two inputs.

SOME MATRIX MEASURES

Determinant:

$$\left|[A]\right| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

$$\left|[A]\right| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{32} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Etc.

Exists only for square matrices.

Determinant is zero for singular matrices.

$$\left|[A][B]\right| = \left|[A]\right|\left|[B]\right| \qquad \left|[A]\right| = \left|[A]^T\right| \qquad \left|[A]^*\right| = \left|[A]^H\right| \qquad \left|k[A]\right| = k^n\left|[A]\right|$$

Trace:

Sum of diagonal elements
for a square matrix.

$$tr([A]) = \sum_i a_{ii}$$

VECTOR SPACE

The set of all n dimensional vectors make up an n dimensional vector space, R^n.

n vectors $\{e\}_i$ in R^n are said to be ***linearly independent*** if the equation

$$0 = a_1\{e\}_1 + a_2\{e\}_2 + a_3\{e\}_3 \cdots a_n\{e\}_n$$

only has a solution in which the constants a_i are all zero. The n vectors $\{e\}_i$ are said to ***span*** the vector space.

Any n dimensional vector $\{x\}$ can be expressed uniquely as a linear combination of the n linearly independent vectors $\{e\}_i$:

$$\{x\} = b_1\{e\}_1 + b_2\{e\}_2 + b_3\{e\}_3 \cdots b_n\{e\}_n$$

VECTOR SPACE THEORY APPLIED TO MATRICES

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1}$$

Rank of a matrix is the number of linear independent columns or rows.

$$rk([A])=r$$

$rk([A]) = m$ $[A]$ is full column rank.

$rk([A]) = r < m,n$ $[A]$ is rank deficient. If $[A]$ square, it is **singular** and $[A]^{-1}$ does not exist.

Column Space of $[A]$ is the vector space spanned by its columns. $R([A])$

Row Space of $[A]$ is the vector space spanned by its rows.

Null Space of $[A]$ is the set of vectors $\{x\}$ such that: $[A]\{x\}=\{0\}$

$$N([A])$$

EXAMPLE

$$[A] = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 7 \\ 8 & 3 & -13 \\ -1 & 4 & 6 \end{bmatrix}$$

$$rk([A]) = 2$$

$$\left| [A]^T [A] \right| = 0$$

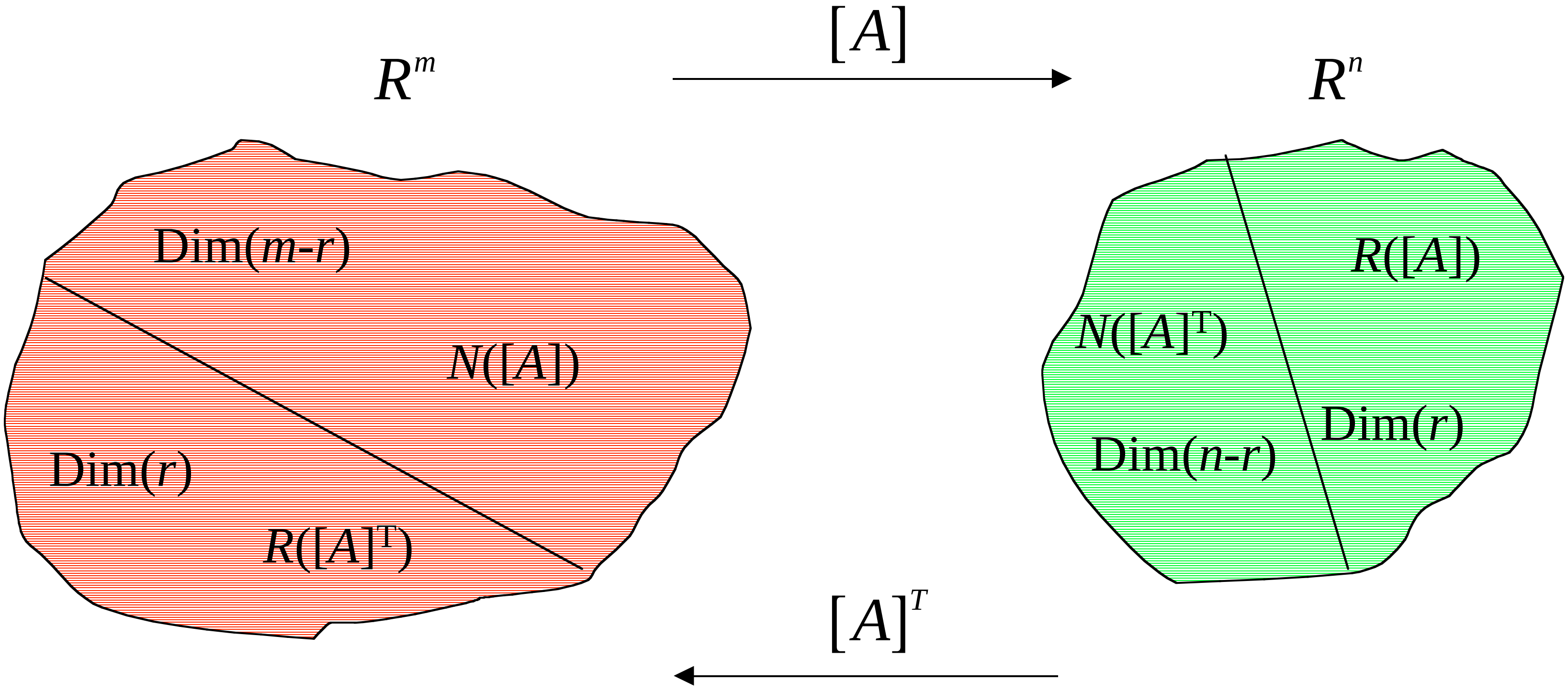
Determinant of any 2-by-2 submatrix is nonzero.

$$R([A]) = \begin{bmatrix} 0.12 & 0.37 \\ -0.35 & 0.18 \\ 0.92 & 0.55 \\ -0.12 & 0.73 \end{bmatrix}$$

$$N([A]) = \begin{bmatrix} 0.82 \\ -0.41 \\ 0.41 \end{bmatrix}$$

FOUR SUBSETS OF MATRIX ALGEBRA

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1} \qquad [A] \text{ has rank } r$$



R^m

$\xrightarrow{[A]}$

R^n

Dim(m-r)

$N([A])$

Dim(r)

$R([A]^{\mathrm{T}})$

$R([A])$

$N([A]^{\mathrm{T}})$

Dim(r)

Dim(n-r)

$\xleftarrow{[A]^T}$

SPECTRAL DECOMPOSITION

In most cases, a square *n*-by-*n* matrix [*A*] can be decomposed into a product of three matrices

$$[A] = [\Phi][\Omega][\Phi]^{-1}$$

where [Ω] is diagonal with entries λ_i called **eigenvalues** and [Φ] is the modal matrix containing columns $\{\phi\}_i$ called **eigenvectors**.

This implies that [*A*] is **diagonalizable**.

$$[\Phi]^{-1}[A][\Phi] = [\Omega]$$

Eigenvalues and eigenvectors satisfy the eigen-problem:

$$([A] - \lambda_i [I])\{\phi\}_i = \{0\}$$

Eigenvalues satisfy the characteristic equation:

$$\left|([A] - \lambda_i [I])\right| = 0$$

Note: Eigenvectors $\{\phi\}_i$ are in null space of matrix $([A] - \lambda_i [\,I\,])$

$$\left|[A]\right| = \prod_{i=1}^{n} \lambda_i$$

SPECTRAL DECOMPOSITION

Matrix $[A]$ is diagonalizable iff it possesses n linearly independent eigenvectors.

If $[A]$ has distinct eigenvalues, it is diagonalizable.

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}^{-1} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad [\Phi] = \begin{bmatrix} 0.23 & 0.79 & 0.41 \\ 0.53 & 0.09 & -0.82 \\ 0.82 & -0.61 & 0.41 \end{bmatrix} \qquad [\Omega] = \begin{bmatrix} 16.12 & 0 & 0 \\ 0 & -1.12 & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

Note: Real nonsymmetric matrices can have complex eigenvalues and eigenvectors. If so, they occur in complex conjugate pairs.

SPECIAL CASE
REAL SYMMETRIC MATRICES

Always are diagonalizable.

Always possess real eigenvalues and eigenvectors.

Modal matrix is orthogonal.

$$[\Phi]^T = [\Phi]^{-1} \qquad \therefore \qquad [\Phi]^T [A][\Phi] = [\Omega]$$

STRUCTURAL DYNAMICS

$$[M]\{\ddot{x}\}+[K]\{x\}=\{0\} \qquad \text{Assume:} \quad \{x\}=\{\phi\}e^{i\omega t}$$

$[M]$ real symmetric **positive definite**. $[K]$ real symmetric **positive semidefinite**.

Multiplying through by $[M]^{-1}$ and substituting leads to eigenvalue problem:

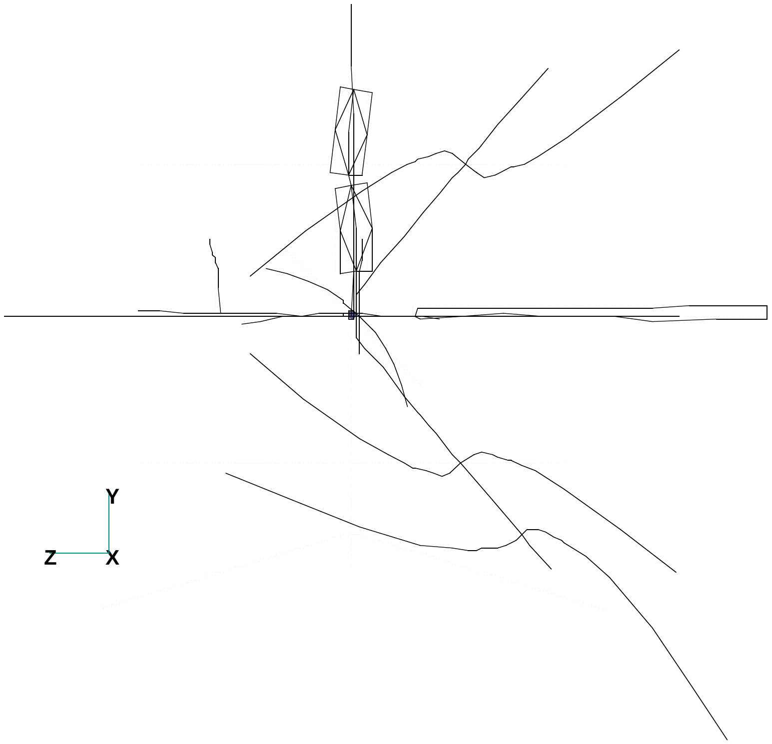
$$\left([M]^{-1}[K]-\lambda_i[I]\right)\{\phi\}_i=\{0\}$$

Modal matrix simultaneously diagonalizes $[M]$ and $[K]$.

$$[\Phi]^T[M][\Phi]=[I] \qquad [\Phi]^T[K][\Phi]=[\Omega]$$

Decoupled system in modal coordinates.

$$\{\ddot{q}\}+[\Omega]\{q\}=[\Phi]^T\{F\}$$



SINGULAR VALUE DECOMPOSITION

Any n x m matrix $[A]$ of rank r can be decomposed into a product of three matrices:

$$[A]=[U][S][V]^H$$

$$[U](n\times n)$$
$$[S](n\times m)$$
$$[V](m\times m)$$

$[U]$ and $[V]$
are Unitary

$$[U]^H[U]=[I]$$
$$[V]^H[V]=[I]$$

Matrices can be partitioned as:

$$[A]=\begin{bmatrix}[U_1] & [U_2]\end{bmatrix}\begin{bmatrix}[\Sigma] & [0]\\ [0] & [0]\end{bmatrix}\begin{bmatrix}[V_1]^H\\ [V_2]^H\end{bmatrix}$$

$$[U_1](n\times r)\quad [U_2](n\times(n-r))$$
$$[V_1](m\times r)\quad [V_2](m\times(m-r))$$

or $\quad [A]=[U_1][\Sigma][V_1]^H$ ⇒ $[A]=\sum_{i=1}^{r}\{u_1\}_i\sigma_i\{v_1\}_i^H$

$\{u_1\}_i$ is ith
column of $[U_1]$

SINGULAR VALUE DECOMPOSITION

$$[\Sigma](r \times r) = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots \\ 0 & \sigma_2 & 0 & \cdots \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \sigma_r \end{bmatrix}$$

Singular values are
real and satisfy:

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

$\sigma_i^2, \{u_1\}_i$ - are eigenvalues and eigenvectors of $[A][A]^T$

$\{u_2\}_i$ - are eigenvectors of $[A][A]^T$ with zero eigenvalues.

$\sigma_i^2, \{v_1\}_i$ - are eigenvalues and eigenvectors of $[A]^T[A]$

$\{v_2\}_i$ - are eigenvectors of $[A]^T[A]$ with zero eigenvalues.

$[U_1]$ spans the column space of $[A]$.

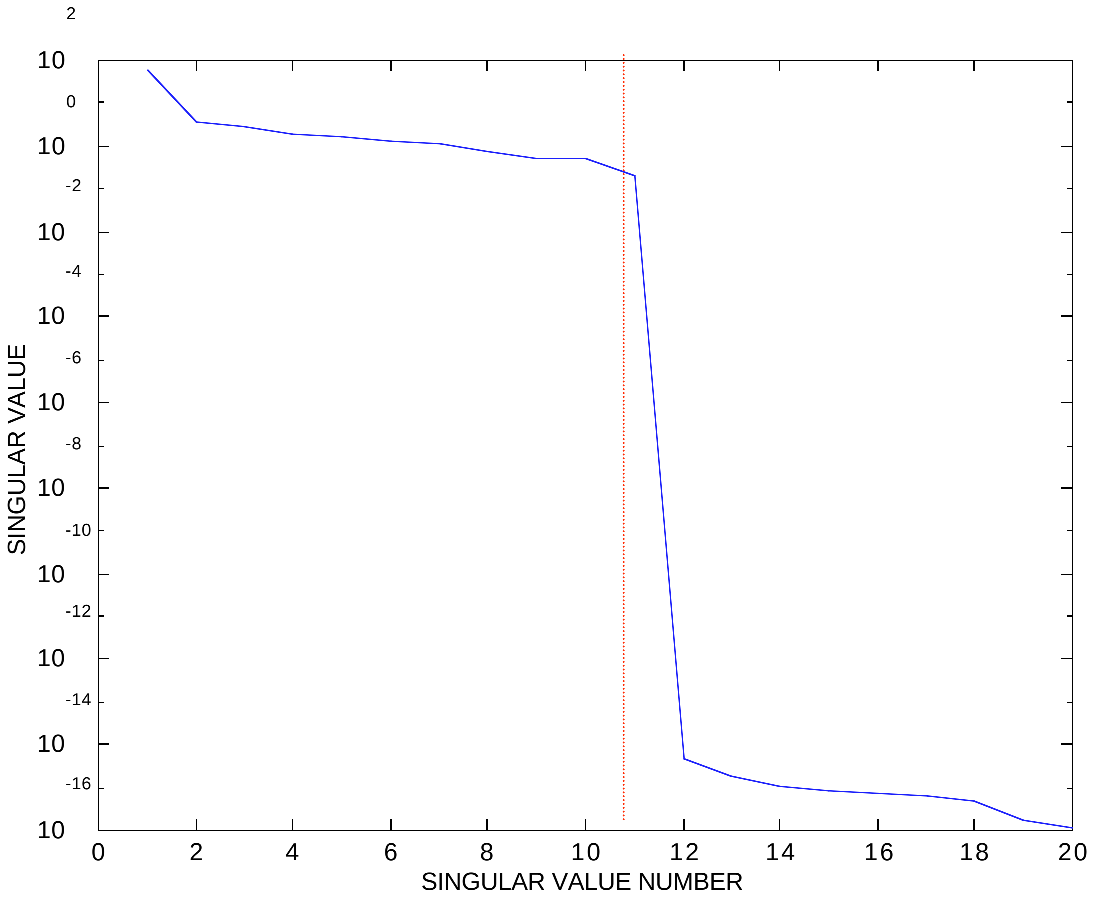
$[U_2]$ spans the column null space of $[A]$.

$[V_1]^T$ spans the row space of $[A]$.

$[V_1]^T$ spans the row null space of $[A]$.

SVD CAN BE USED TO DETERMINE RANK OF A MATRIX

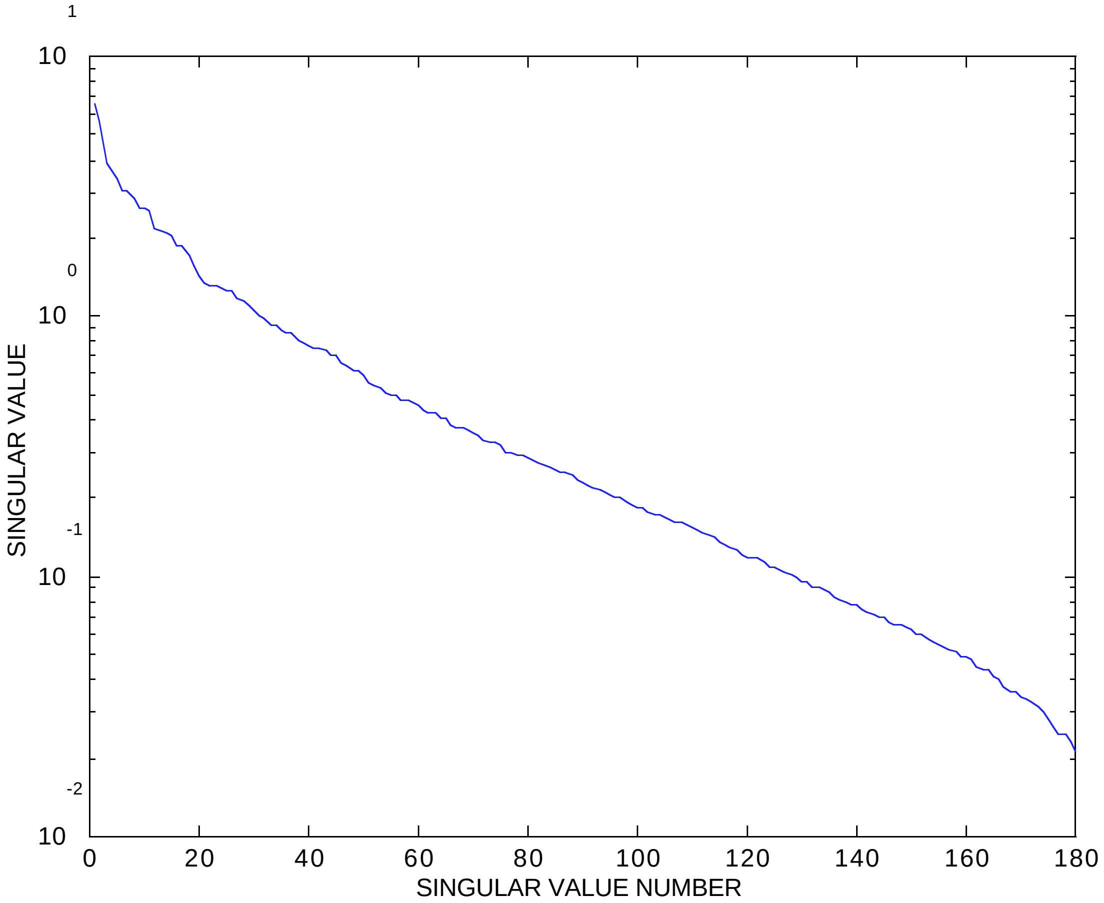
It is obvious to the casual observer that matrix [*A*] has a rank of 11



UNFORTUNATELY, IT IS NOT ALWAYS SO EASY

Singular values for a block Hankel matrix 640 rows and 1380 columns.

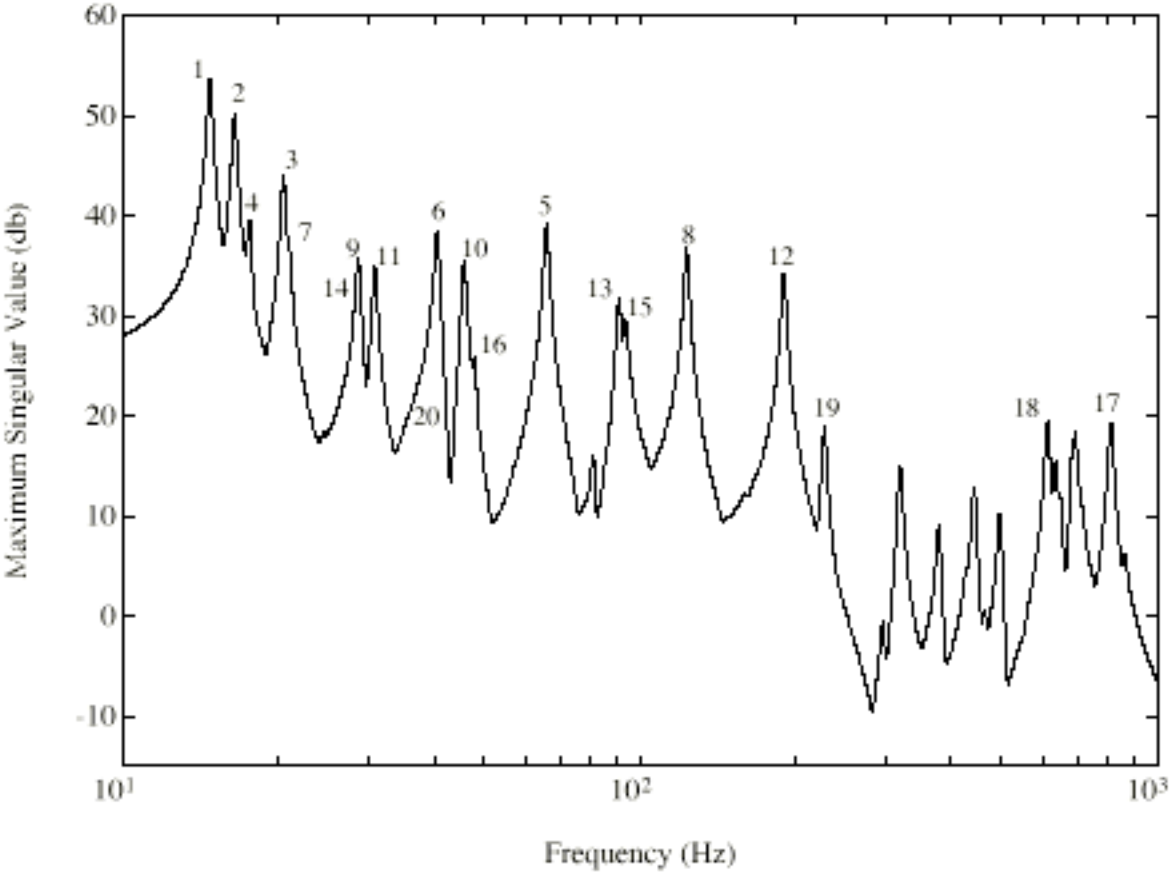
What’s the rank?



MAXIMUM SINGULAR VALUE OF FRF MATRIX VS. FREQUENCY

Gives measure of participation of modes in system response.

Use to rank dynamic importance of modes.



SOLUTION OF DETERMINED EQUATIONS

Have as many independent equations as unknowns.

$$[A]_{n\times n}\{x\}_{n\times 1}=\{b\}_{n\times 1}$$

$[A]$ is square, full rank, and **invertible**.

$$\{x\}=[A]^{-1}\{b\}$$

$\{b\}$ is **always** in range space of $[A]$.

A **unique** solution **always** exists.

MATRIX INVERSE

The inverse of a nonsingular matrix $[A]$ is a matrix $[A]^{-1}$ that when multiplied by $[A]$ is the identity matrix.

$$[A][A]^{-1} = [A]^{-1}[A] = [I]$$

Properties:

$$([A][B])^{-1} = [B]^{-1}[A]^{-1}$$

$$\left([A]^T\right)^{-1} = \left([A]^{-1}\right)^T$$

$$\left([A]^H\right)^{-1} = \left([A]^{-1}\right)^H$$

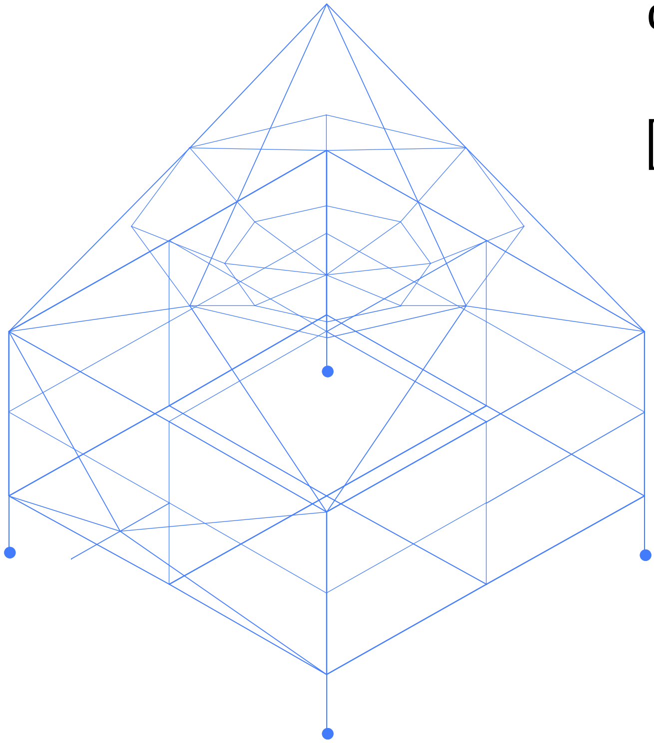
$$(k[A])^{-1} = \frac{1}{k}[A]^{-1}$$

$$\left|[A]^{-1}\right| = \frac{1}{\left|[A]\right|}$$

Application:

Solve for static deflection

$$[K]\{x\} = \{F\}$$



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A METHOD TO COMPUTE INVERSE

$$[A]^{-1} = \frac{[Adjoint([A])]}{\|[A]\|}$$

Adjoint of $[A]$ is a matrix with elements equal to the cofactors of $[A]$ transposed.

Let $[M_{ij}]$ be the submatrix of $[A]$ obtained by deleting the ith row and jth column. The determinant of $[M_{ij}]$ is called a **minor** of $[A]$.

$$Cofactor([A])_{ij} = c_{ij} = (-1)^{i+j}\left\|[M_{ij}]\right\|$$

$$[A] = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \Longrightarrow \quad [A]^{-1} = \frac{1}{\|[A]\|}\begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ +\begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

There are better methods!

LU DECOMPOSITION

Any nonsingular matrix square matrix $[A]$ can be factored into the product of two matrices.

$$[A]=[L][U]$$

$[L]$ is a **lower triangular** matrix and $[U]$ is an **upper triangular** matrix.

$$[A]=\begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix}$$

⇨

$$[L]=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -4 & 2 & 1 & 0 \\ -2 & -1 & -2 & 2 \end{bmatrix}$$

$$[U]=\begin{bmatrix} 3 & -1 & -2 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

LU DECOMPOSITION
WHAT'S IT GOOD FOR?

Inverse:
$$[A]^{-1} = ([L][U])^{-1} = [U]^{-1}[L]^{-1}$$

Determinant:
$$\|[A]\| = \|[L]\|\|[U]\|$$

Solving determined set of equations.
$$[A]\{x\} = \{b\} \quad \Rightarrow \quad [L][U]\{x\} = \{b\}$$

Can be written as two sets of equations.

$$[L]\{z\} = \{b\}$$

Solve using **forward substitution**

$$[U]\{x\} = \{z\}$$

Solve using **backward substitution**

BACKWARD SUBSTITUTION

$$[U]\{x\}=\{z\} \quad \Longrightarrow \quad \begin{bmatrix} 3 & -1 & -2 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 5 \\ 2 \\ -6 \end{Bmatrix}$$

Last equation implies:

$$x_4 = \frac{-6}{4}$$

Substitute into third equation, etc.

TEST-ANALYSIS-MODEL DEVELOPMENT USING STATIC REDUCTION

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$$

n dof in FEM

Objective: Reduce mass matrix to test sensor locations for test-analysis correlation and still maintain accurate modal properties.

Partition static equation into sensor dof (a-set) and dof to be reduced out (o-set).

$$\begin{bmatrix}[K_{oo}] & [K_{oa}]\\ [K_{ao}] & [K_{aa}]\end{bmatrix}\begin{Bmatrix}\{x_o\}\\ \{x_a\}\end{Bmatrix}=\begin{Bmatrix}\{0\}\\ \{F\}\end{Bmatrix}$$

Solve first equation for $\{x_o\}$.

$$\{x_o\}=[K_{oo}]^{-1}[K_{oa}]\{x_a\}$$

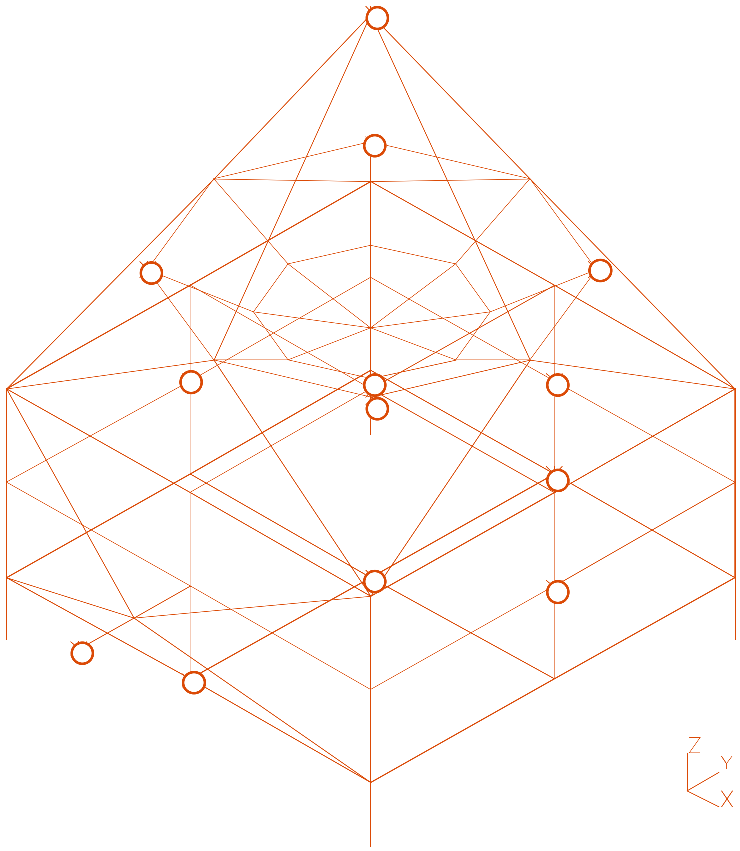
➡

$$\{x\}=\begin{Bmatrix}x_o\\ x_a\end{Bmatrix}=\begin{bmatrix}[K_{oo}]^{-1}[K_{oa}]\\ [I]\end{bmatrix}\{x_a\}=[T]\{x_a\}$$

$$[M]_{TAM}=[T]^T[M][T]$$

$[T]$ is a **transformation** from $\{x_a\}$ to $\{x\}$.

STATIC REDUCTION OFTEN GIVES GOOD RESULTS



300 Degrees of Freedom

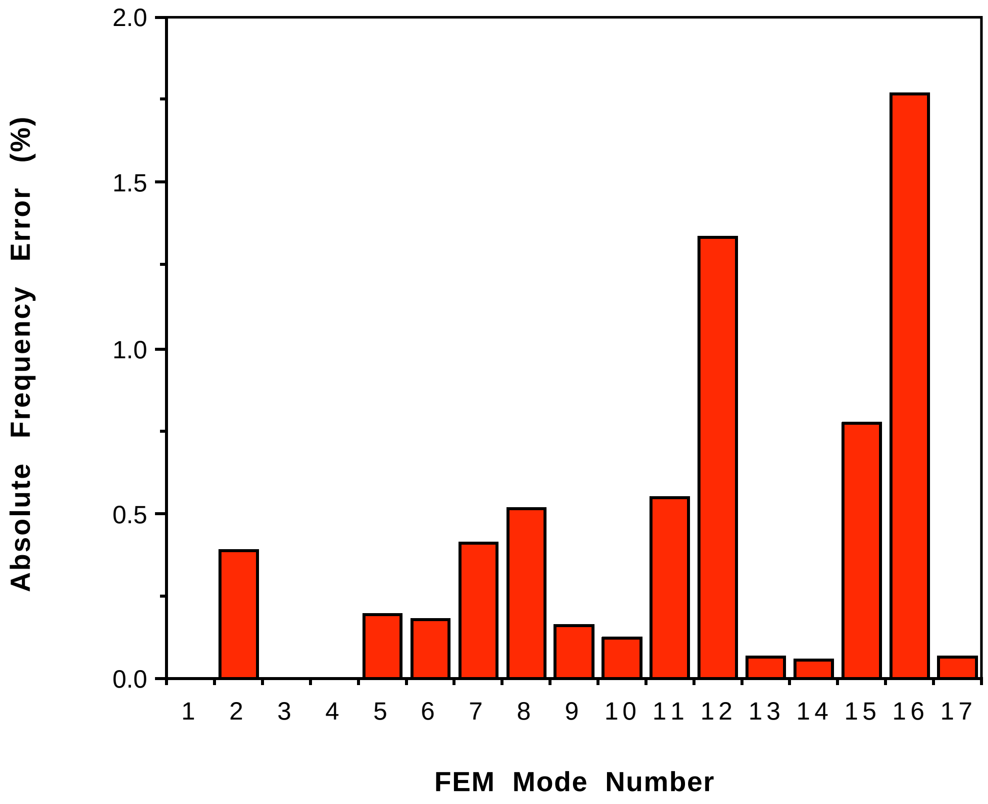
90 Shell and Bar Elements

17 Modes with Frequencies below 70.0 Hz

Want TAM to be able to Predict all 17 Modes Under 70.0 Hz.

○ - denotes master dof location

20 DOF STATIC TAM CAN ACCURATELY REPRESENT GPSC FEM MODAL PARAMETERS IN 0.0 - 70.0 Hz. FREQUENCY RANGE



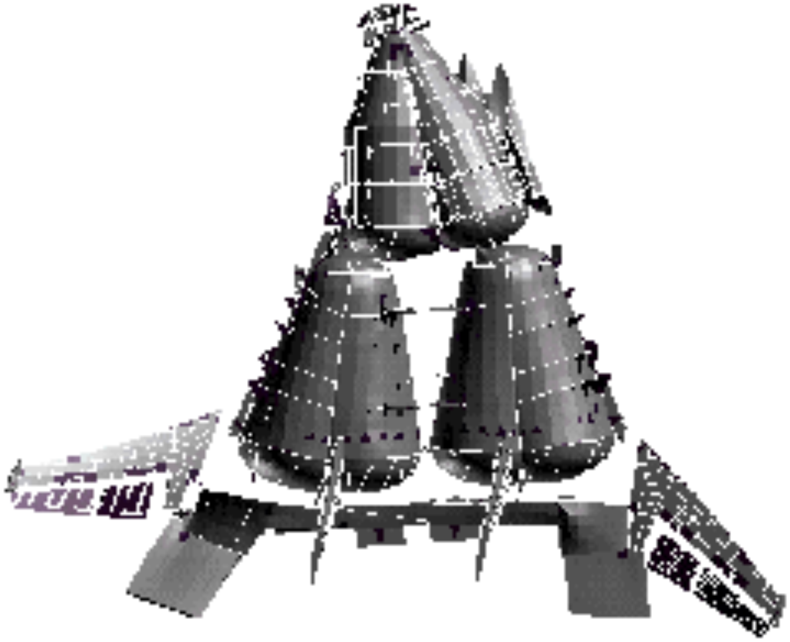
X-33 REPRESENTS A MORE CHALLENGING PROBLEM

100,000 Dof

833 modes
below 55.0 Hz.

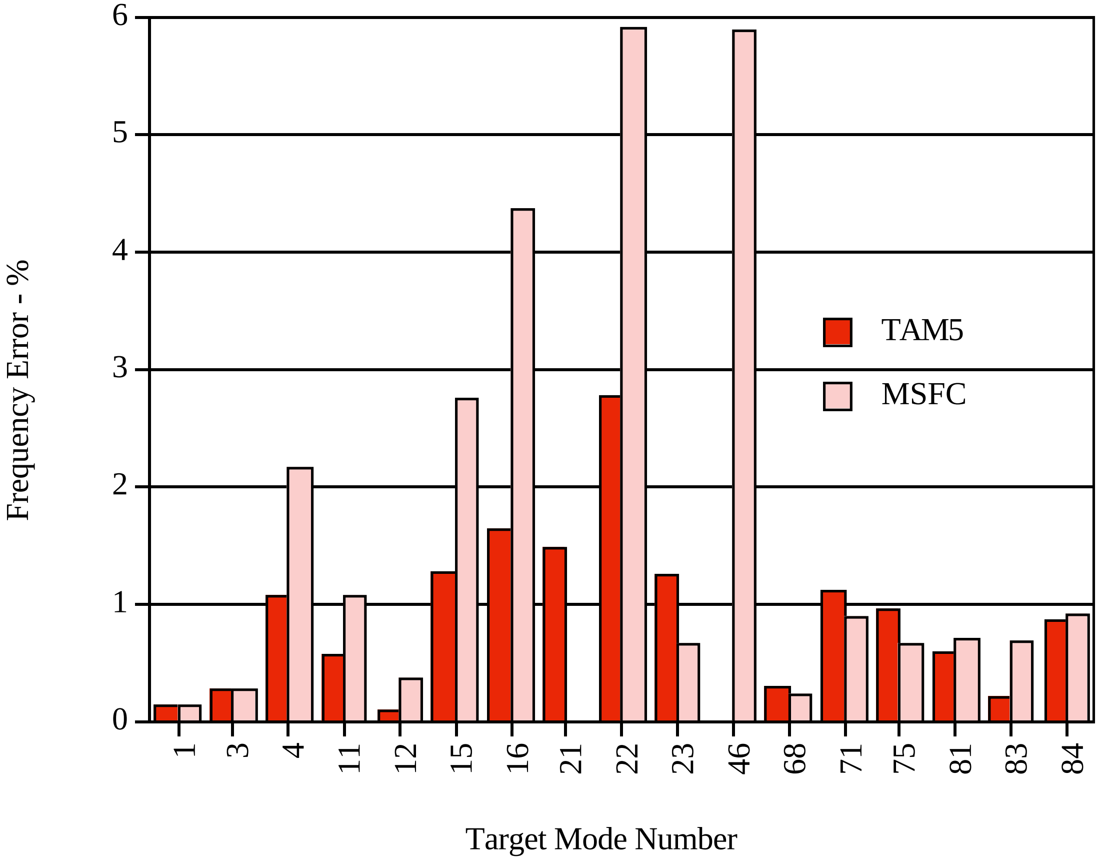
17 TARGET MODES BELOW 25 Hz.

Canted Fin Symmetric Bending (6.41 Hz)



Vehicle Yaw (11.0 Hz)

X-33 FEM/TAM FREQUENCY ERROR



1200 dof static TAMs accurately predict most of the target modes below 25.0 Hz., but two are not predicted and none of the 10 target modes above 25.0 Hz. are predicted accurately.

SOLUTION OF UNDERDETERMINED EQUATIONS

Have **fewer** independent equations than unknowns.

$[A]$ is rectangular with more columns than rows and is assumed to be **full row rank**.

A solution **always** exists, but there are ***infinitely*** many.

$\text{Dim}(N([A])) = m - r = m - n$

$$[A]_{n\times m}\{x\}_{m\times 1} = \{b\}_{n\times 1} \qquad n < m$$

$\{b\}$ is **always** in column space of $[A]$.

Suppose $\{x\}_1$ is a solution, then if $\{x\}_n$ is any vector in $N([A])$,

$$\{x\}_s = \{x\}_1 + \{x\}_n$$

is also a solution.

BRUTE FORCE SOLUTION PROCEDURE FOR UNDERDETERMINED SYSTEMS

$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1}$ $n<m$

Define a new vector: $\{x\}=[A]^T\{z\}$

The original matrix Equation becomes: $[A][A]^T\{z\}=\{b\}$

$[A]$ is full row rank, therefore $[A][A]^T$ is nonsingular.

Solve for $\{z\}$:

$$\{z\}=\left([A][A]^T\right)^{-1}\{b\} \quad \Rightarrow \quad [A]^T\{z\}=[A]^T\left([A][A]^T\right)^{-1}\{b\}=\{x\}_m$$

$$[A]^+=[A]^T\left([A][A]^T\right)^{-1}$$

Right Generalized Inverse of $[A]$

Out of infinity, which solution is $\{x\}_m$?

For arbitrary solution $\{x\}_s$: $\{b\}=[A]\{x\}_s$

Substitute: $\Rightarrow$

$$\{x\}_m=[A]^T\left([A][A]^T\right)^{-1}[A]\{x\}_s=[P]\{x\}_s$$

PROJECTORS

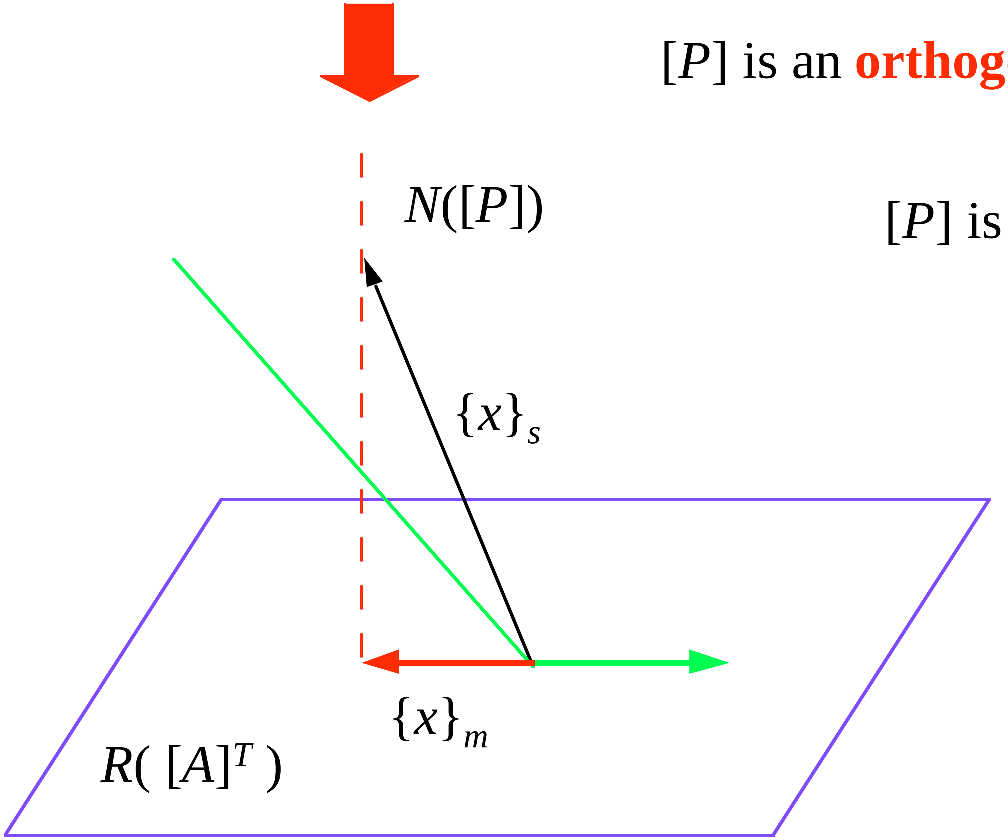
$$\{x\}_m = [P]\{x\}_s$$

$$[P]_{m\times m} = [A]^T\left([A][A]^T\right)^{-1}[A]$$

Is the **orthogonal projection** of general solution $\{x\}_s$ onto the row space of $[A]$ or column space of $[A]^T$.

$\{x\}_m$ is the **minimum norm** (length) solution.

$[P]$ is an **orthogonal projector**.



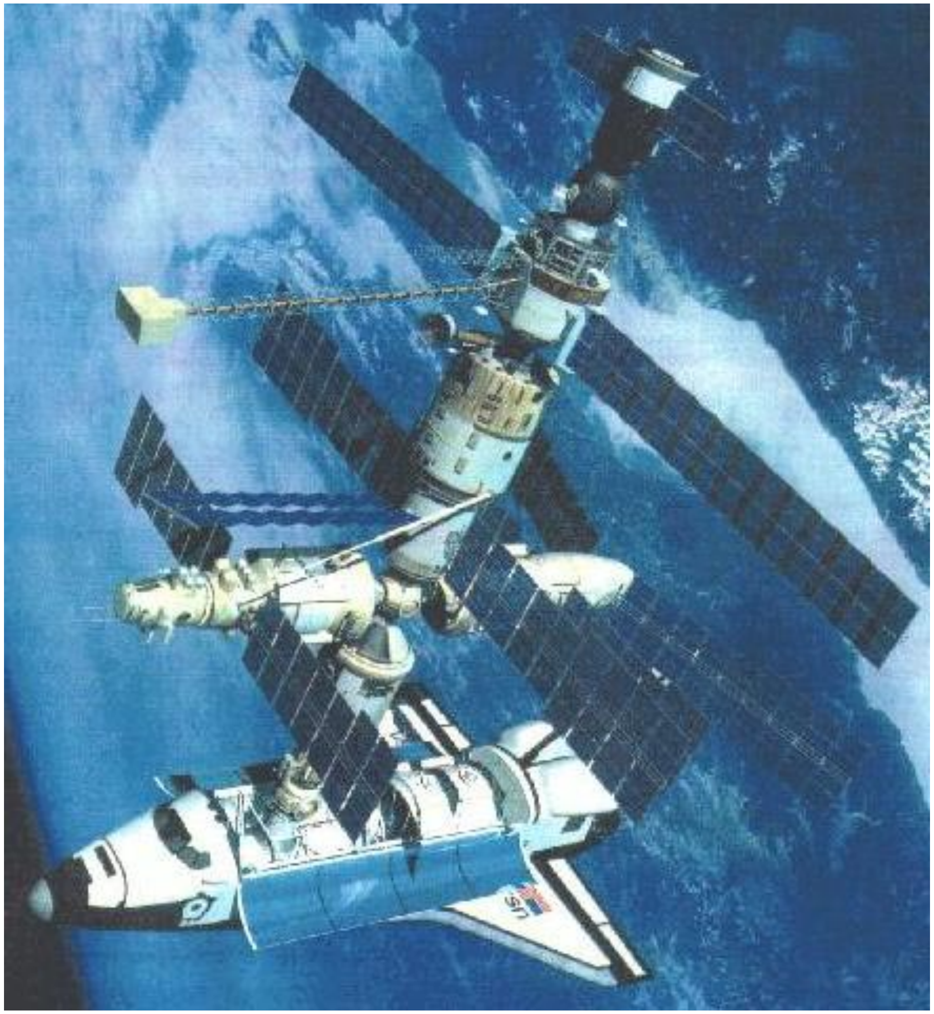
$[P]$ is Idempotent: $[P][P] = [P]$

$$tr([P]) = rk([P]) = r = n$$

r eigenvalues of magnitude 1.0

m-r eigenvalues of magnitude 0.0

APPLICATION TO TIME DOMAIN SYSTEM IDENTIFICATION



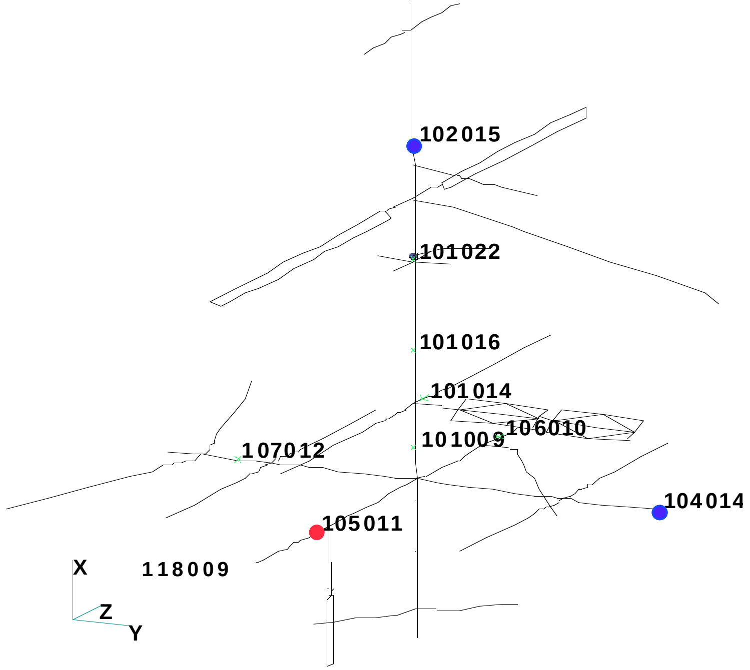
Identify MIR space station modal parameters using response from excitation due to Space Shuttle docking.

IDENTIFY SYSTEM MARKOV PARAMETERS USING ONE DATA SET

Measure input and output and form underdetermined matrix equation.

$$[Y_S][H]=[Y_D]$$

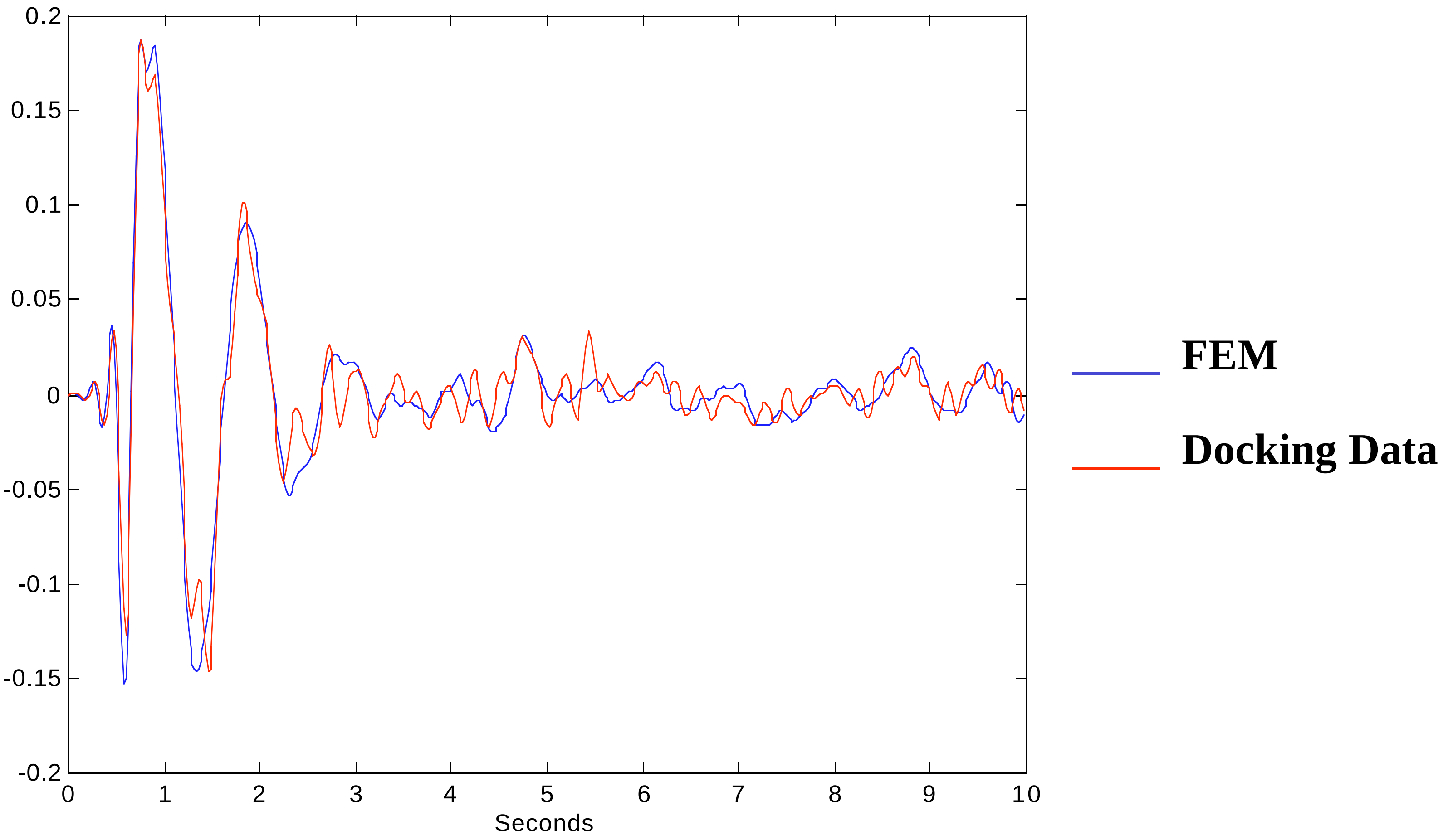
Solution process projects [H] onto Row space of data set.



$$\begin{bmatrix} y_{s0} & 0 & \cdots & \cdots & 0 \\ y_{s1} & y_{s0} & 0 & \cdots & 0 \\ y_{s2} & y_{s1} & y_{s0} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_{sn_t-1} & y_{sn_t-2} & \cdots & \cdots & y_{sn_t-N_R} \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{N_R} \end{bmatrix} = \begin{bmatrix} y_{d0} \\ y_{d1} \\ \vdots \\ y_{dn_t-1} \end{bmatrix}$$

[Y_s] has 650 rows and 1950 columns in application.

PULSE RESPONSE AT 104014z PROJECTED ONTO STS-81 DATA



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SOLUTION OF OVERDETERMINED EQUATIONS

Have **more** independent equations than unknowns.

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1} \qquad n>m$$

$[A]$ is rectangular with more rows than columns and is assumed to be **full column rank**.

$\{b\}$ **may or may not** be in column space of $[A]$.

If $\{b\}$ is in $R([A])$:

A **unique** solution exists.

If $\{b\}$ is not in $R([A])$:

No exact solution exists. But can find an approximate one.

BRUTE FORCE SOLUTION PROCEDURE FOR OVERDETERMINED SYSTEMS

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1} \qquad n>m$$

Premultiply by $[A]^T$

$$\left([A]^T[A]\right)\{x\}=[A]^T\{b\}=\{\bar{b}\}$$

$([A]^T[A])$ is nonsingular, therefore equation can be **solved using previously discussed techniques**.

⇒ $$\{x\}=\left([A]^T[A]\right)^{-1}[A]^T\{b\} \qquad [A]^{+}=\left([A]^T[A]\right)^{-1}[A]^T$$

Left Generalized Inverse of $[A]$

Plug this solution into original equation:

$$[A]\{x\}=[A]\left([A]^T[A]\right)^{-1}[A]^T\{b\}=\left[\hat{P}\right]\{b\}=\left\{\hat{b}\right\}$$

$\left[\hat{P}\right]$ is an orthogonal projector onto $R([A])$ ⇒ $\left\{\hat{b}\right\}$ is the orthogonal projection of $\{b\}$ onto $R([A])$

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BRUTE FORCE SOLUTION PROCEDURE FOR OVERDETERMINED SYSTEMS

Case I: $\{b\} \in R([A])$ ⇒ $\{x\} = \left([A]^T[A]\right)^{-1}[A]^T\{b\}$

$\left[\hat{P}\right]\{b\} = \{b\}$ ⇒ Unique solution to $[A]\{x\}=\{b\}$

Case II: $\{b\} \notin R([A])$ ⇒ $\{\hat{x}\} = \left([A]^T[A]\right)^{-1}[A]^T\{b\}$

$\left[\hat{P}\right]\{b\} = \left\{\hat{b}\right\}$ ⇒ Unique solution to $[A]\{x\}=\left\{\hat{b}\right\}$

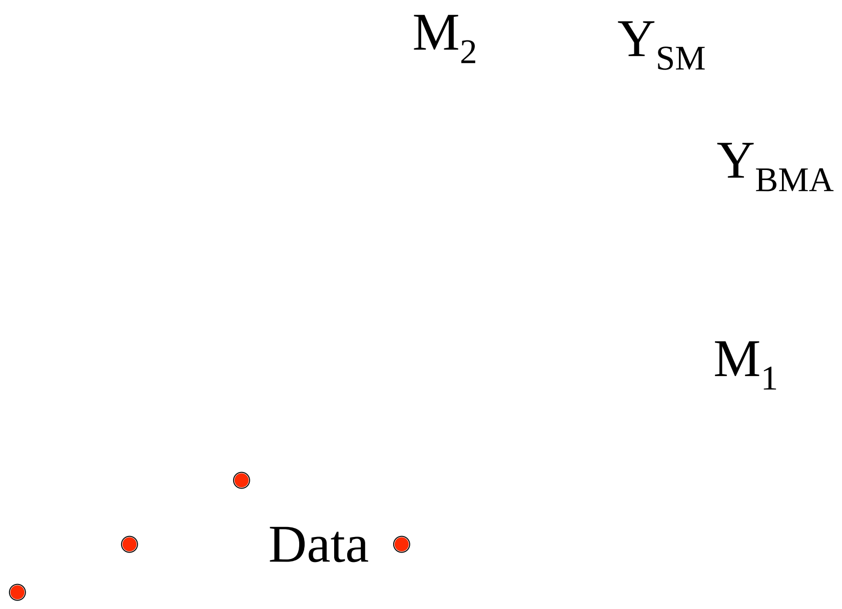
$\{\hat{x}\}$ Minimizes Euclidean norm (length) of error vector $\{e\}$.

$\{e\} = \{b\} - [A]\{x\}$

Least-Squares Solution

Works well in the presence of noise.

CURVE FITTING



PREVIOUS BRUTE FORCE TECHNIQUE FOR LEAST-SQUARES USES THE NORMAL FORM

$$\left([A]^T[A]\right)\{x\} = [A]^T\{b\}$$

Direct inversion of ($[A]^T$ $[A]$) is ***costly*** and ***inaccurate***: $o\left(n^3\right)$

LU decomposition of ($[A]^T$ $[A]$) and backward/forward substitution is ***faster*** but can still be ***inaccurate***: $o\left(n^2\right)$

In general, it's best not to use the Normal Form of the equations to obtain the Least-Squares solution.

SOLUTION OF LEAST-SQUARES PROBLEM USING QR DECOMPOSITION

Operates directly on matrix equation for general full column rank matrix:

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1} \qquad n>m$$

$$[Q]^T[Q]=[Q][Q]^T=[I]$$

$[A]$ can be uniquely factored in the form:

$$[A]_{n\times m}=[Q]_{n\times n}[R]_{n\times m}$$

$[R]$ upper triangular

$$[R]=\begin{bmatrix}[U]\\ [0]\end{bmatrix}$$

$$[Q][R]\{x\}=\{b\}$$

Premultiply by $[Q]^T$

$$[Q]^T[Q][R]\{x\}=[R]\{x\}=[Q]^T\{b\}$$

Partition:

$$\begin{bmatrix}[U]\\ [0]\end{bmatrix}\{x\}=\begin{bmatrix}[Q_1]^T\\ [Q_2]^T\end{bmatrix}\{b\} \quad ➡ \quad [U]\{x\}=[Q_1]^T\{b\}$$

Solve by backsubstitution.

Preferred Technique for L.S.
Fast and Accurate

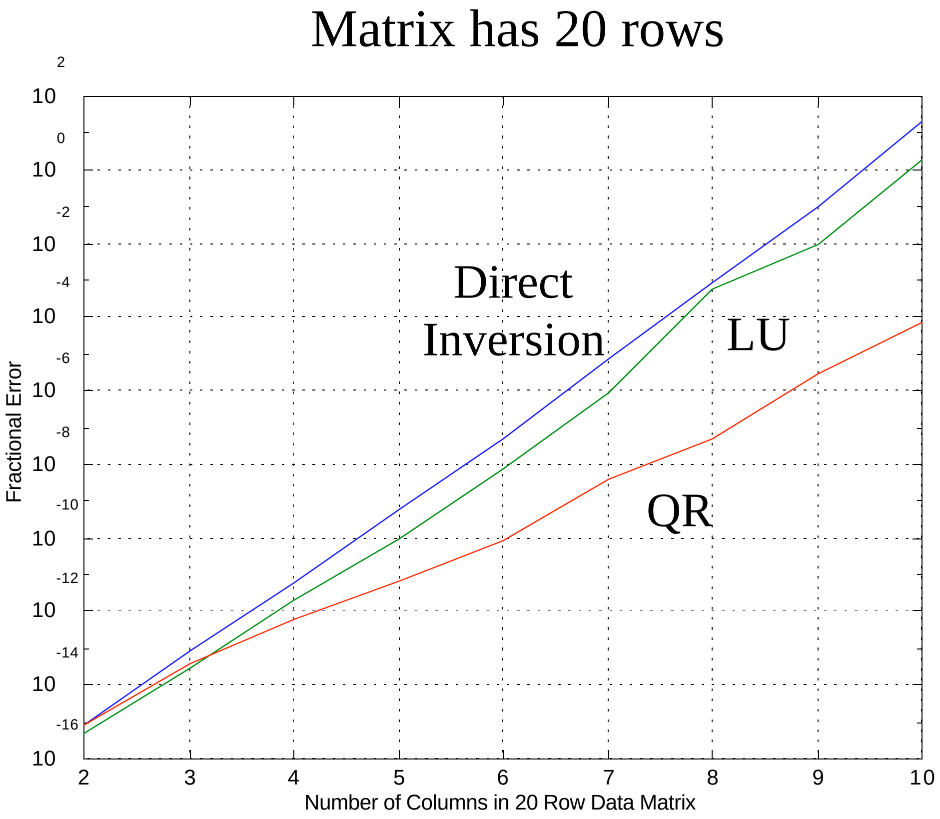
$$o\left(n^2\right)$$

EXAMPLE - POLYNOMIAL CURVE FITTING (JUNKINS)

$$\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^{n-1} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{Bmatrix}$$

$$\{t_1 \quad t_2 \quad \cdots \quad t_m\} = \{0 \quad 1 \quad 2 \quad \cdots \quad m-1\}$$

$$\{y_1 \quad y_2 \quad \cdots \quad y_m\} = \{1 \quad 1 \quad 1 \quad \cdots \quad 1\}$$


Matrix has 20 rows

MOORE-PENROSE GENERALIZED INVERSE

For EVERY matrix $[A]_{nxm}$ of rank r, an mxn matrix $[A]^{+}$ is called a Generalized Inverse if it satisfies:

$$[A][A]^{+}[A]=[A]$$

$$[A]^{+}[A][A]^{+}=[A]^{+}$$

Infinitely Many
generalized inverses

If in addition, $[A]^{+}$ satisfies:

$$[A]^{+}[A]=\left([A]^{+}[A]\right)^{H}$$

$$[A][A]^{+}=\left([A][A]^{+}\right)^{H}$$

Then, inverse is UNIQUE and called the Moore-Penrose Generalized Inverse.

$$[A]^{\dagger}$$

M-P INVERSE MOST ACCURATELY CALCULATED USING SVD

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1}$$

$[A]$ has rank r

SVD ⇨

$$[A]=\begin{bmatrix}[U_1] & [U_2]\end{bmatrix}\begin{bmatrix}[\Sigma] & [0]\\ [0] & [0]\end{bmatrix}\begin{bmatrix}[V_1]^H\\ [V_2]^H\end{bmatrix}$$

$$[A]=[U_1][\Sigma][V_1]^H$$

Moore-Penrose
Inverse given by:

$$[A]^{\dagger}=[V_1][\Sigma]^{-1}[U_1]^H$$

$$[\Sigma]^{-1}(r\times r)=\begin{bmatrix}\sigma_1^{-1} & 0 & 0 & \cdots\\ 0 & \sigma_2^{-1} & 0 & \cdots\\ 0 & 0 & \ddots & 0\\ \vdots & \vdots & \vdots & \sigma_r^{-1}\end{bmatrix}$$

SVD is costly but very
Accurate and Stable.

Trick is to determine correct
value of r.

How small is small, when it
comes to singular values?

MATLAB truncates singular values at:

$$tol=max(n,m)*\sigma_1*2.22\times 10^{-16}$$

MOORE-PENROSE GENERALIZED INVERSE CAN SOLVE MATRIX EQUATIONS FOR ANY CASE

$$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1} \quad ⇨ \quad \{\hat{x}\}=[A]^{\dagger}\{b\}$$

The following equivalencies can be derived.

Case I: Minimum norm: $n < m$ and r = rank([A]) = n.

$$[A]^{\dagger}=[A]^{T}\left([A][A]^{T}\right)^{-1}$$

Case II: Least squares: $n > m$ and r = rank([A]) = m.

$$[A]^{\dagger}=\left([A]^{T}[A]\right)^{-1}[A]^{T}$$

Case III: General or rank deficient case: r = rank([A])   min(n, m).

$$[A]^{\dagger}=[V_1][\Sigma]^{-1}[U_1]^{H}$$

RANK DEFICIENT EXAMPLE

$$[A]=\begin{bmatrix}1&2&3&0&0\\0&0&4&5&6\\0&0&0&0&7\\1&2&7&5&6\end{bmatrix}\qquad \{x\}=\begin{Bmatrix}1\\1\\1\\1\\1\end{Bmatrix}\qquad \{b\}=\begin{Bmatrix}6\\15\\7\\21\end{Bmatrix}\qquad \{\hat{x}\}=[A]^{\dagger}\{b\}=\begin{Bmatrix}0.3209\\0.6419\\1.4651\\0.6279\\1.0000\end{Bmatrix}$$

```
U =
   1.3032e-01  -4.3184e-01   6.8059e-01  -5.7735e-01
   5.9082e-01   6.8625e-02  -5.5937e-01  -5.7735e-01
   3.3749e-01   8.2273e-01   4.5740e-01  -3.5691e-18
   7.2114e-01  -3.6321e-01   1.2122e-01   5.7735e-01
```

Minimum norm

$$\left\|\{x\}\right\|=2.236$$

$$\left\|\{\hat{x}\}\right\|=2.014$$

```
S =
   1.4570e+01            0            0            0            0
            0   5.8276e+00            0            0            0
            0            0   2.9611e+00            0            0
            0            0            0   3.7220e-16            0
```

tol = 1.62e-14

```
V =
   5.8442e-02  -1.3643e-01   2.7078e-01  -6.9696e-01   6.4722e-01
   1.1688e-01  -2.7286e-01   5.4156e-01  -3.7060e-01  -6.9373e-01
   5.3552e-01  -6.1149e-01   2.2047e-01   4.7939e-01   2.4674e-01
   4.5024e-01  -2.5275e-01  -7.3983e-01  -3.8351e-01  -1.9740e-01
   7.0244e-01   6.8494e-01   1.9349e-01  -1.8459e-16  -1.1620e-16
```

MATRIX CONDITIONING

$[A]_{n\times m}\{x\}_{m\times 1}=\{b\}_{n\times 1}$

Condition Number: $\kappa([A])=\dfrac{\sigma_{max}}{\sigma_{min}}$

Matrix $[A]$ may be considered full rank, but it can still be **Ill-conditioned**.

Ill-conditioning of $[A]$ implies that small changes in the elements of $[A]$ and $\{b\}$ can cause very large changes in the computed solution $\{x\}$.

Relative error in $\{x\}$ can be κ times the relative error in the data.

If $\dfrac{1}{\kappa([A])}$ approaches the computer's floating point precision, the matrix is ill-conditioned.

$$1.0 \leq \kappa([A]) \leq \infty$$

Good Bad

Regularization can help.
Look for solution of well-posed problem in neighborhood of ill-posed problem.

GENERAL PURPOSE SPACECRAFT

ESTIMATE LOADS AT INTERFACE WITH LAUNCH VEHICLE USING MEASURED RESPONSE AND MARKOV PARAMETERS

6 Inputs - 6 Outputs

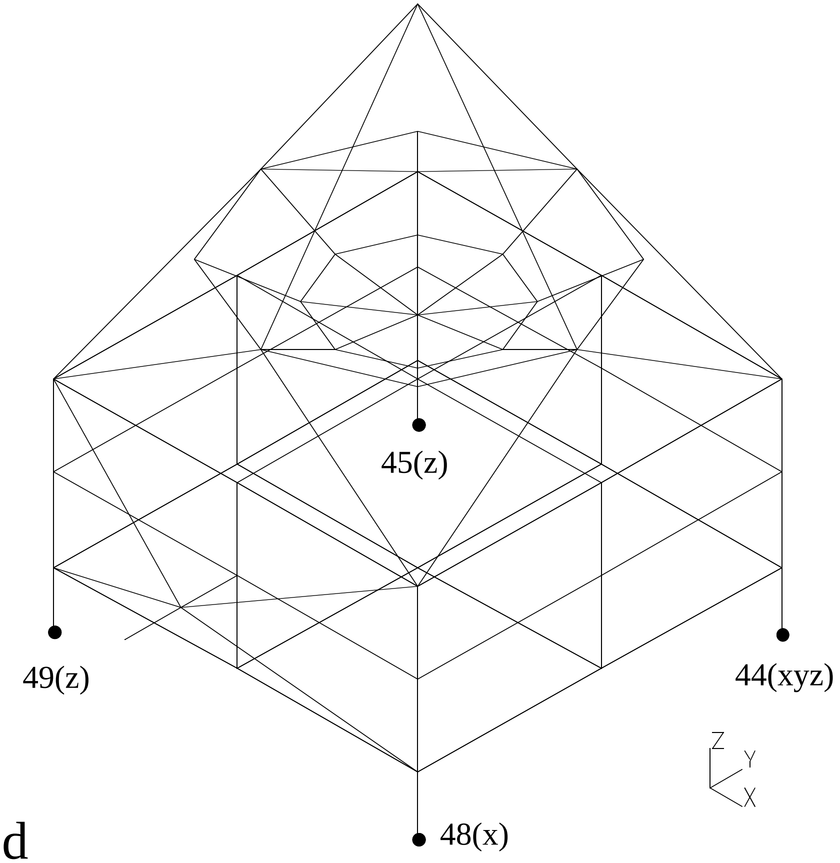
Spacecraft - 0.0-300.0 Hz.

Inputs - 0.0-150.0 Hz.

Inverse problem is ill-posed leading to an ill-conditioned matrix convolution equation.

$$[H]\{U\} = \{Y\}$$

$[H]$ has more rows than columns and quickly becomes ill-conditioned as more data is used.



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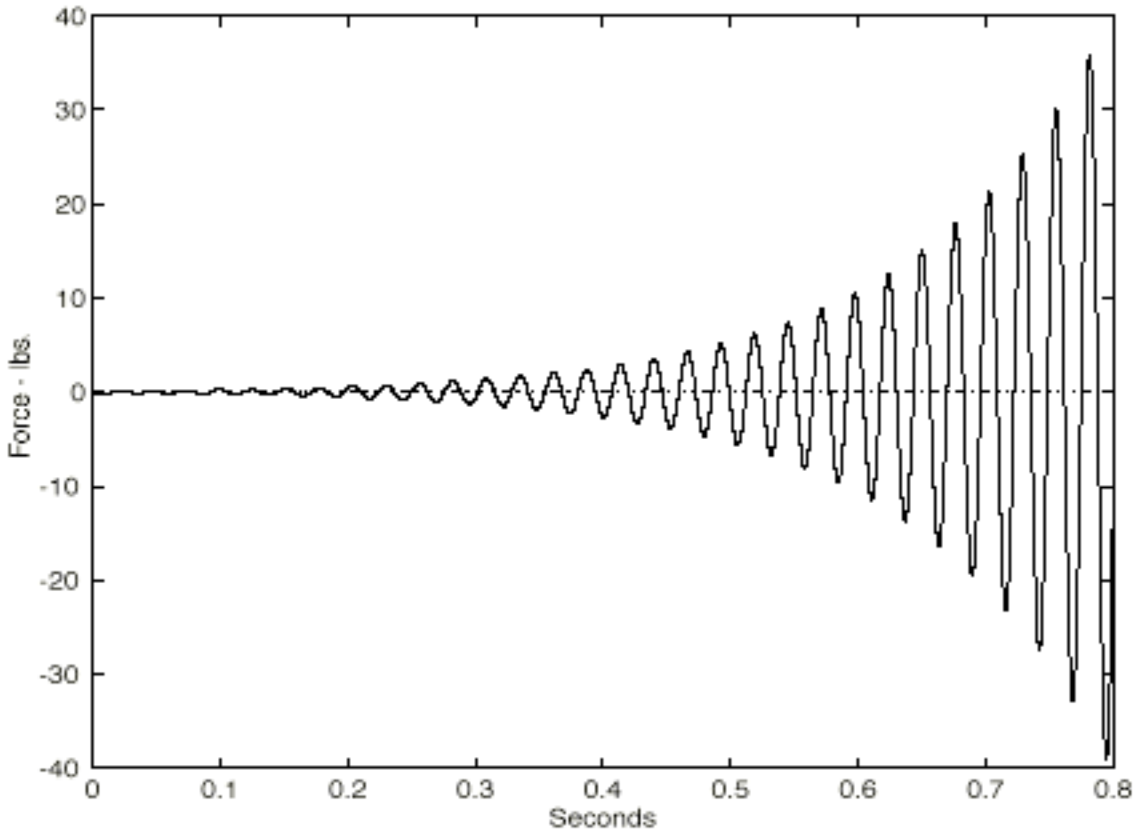
COMPUTE LEAST SQUARES SOLUTION



Ill-conditioning of
data matrix [H] results
in unstable computation
of input forces.

COMPUTATION CAN BE REGULARIZED

Matrix Convolution
Equation

$$[H]\{U\}=\{Y\}$$

$$[H]=\begin{bmatrix} [h_0] & 0 & \cdots & \cdots & 0 \\ [h_1] & [h_0] & 0 & \cdots & \vdots \\ [h_2] & [h_1] & \ddots & 0 & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ [h_{n_t-1}] & [h_{n_t-2}] & \cdots & [h_1] & [h_0] \end{bmatrix}$$

$$\{Y\}=\left\{\{y(0)\}^T \quad \{y(1)\}^T \quad \cdots \quad \{y(n_t-1)\}^T\right\}^T$$

$$\{U\}=\left\{\{u(0)\}^T \quad \{u(1)\}^T \quad \cdots \quad \{u(n_t-1)\}^T\right\}^T$$

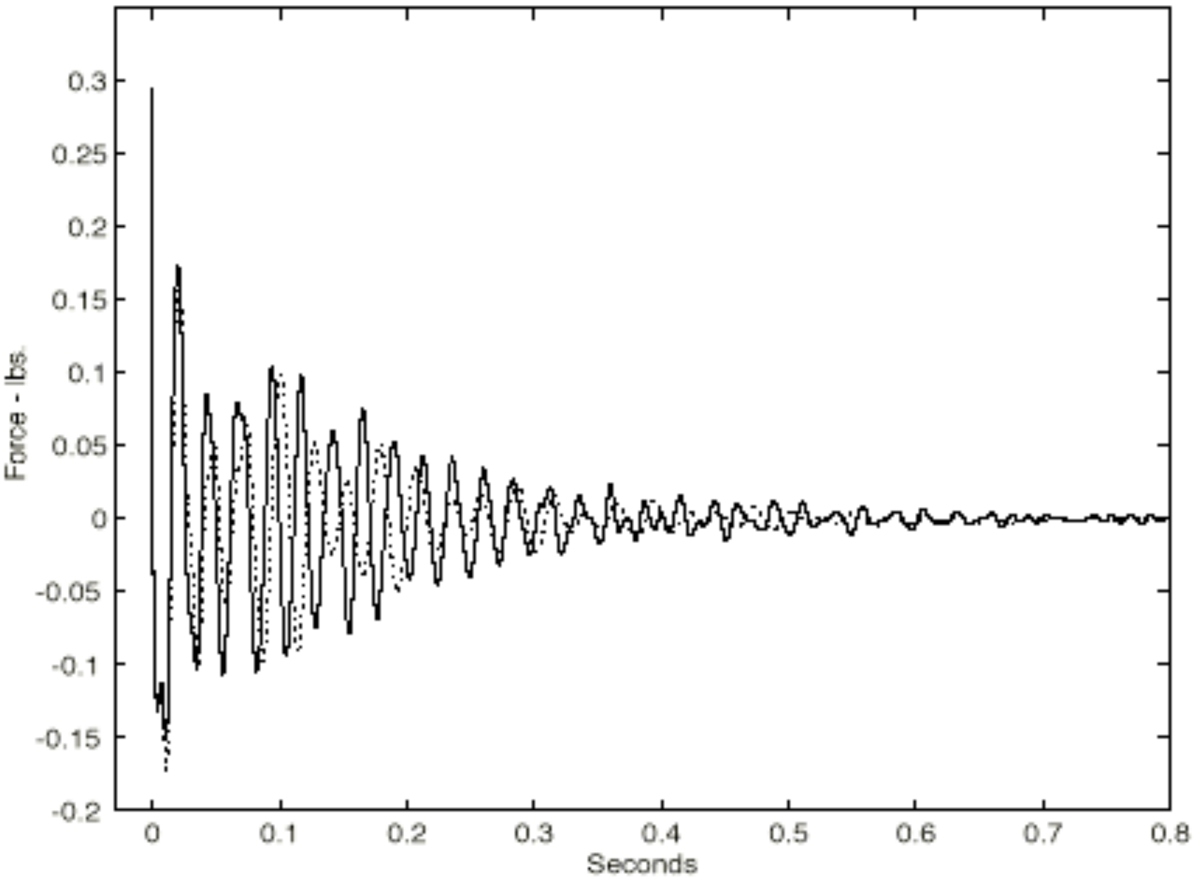
Replace ill-posed problem with closely related well-posed problem.

$$\left([H]^T[H]+\alpha[I]\right)\{U\}=[H]^T\{Y\}$$

Regularized solution:

$$\{U\}_\alpha=\left([H]^T[H]+\alpha[I]\right)^{-1}[H]^T\{Y\}$$

REGULARIZED FORCE COMPUTATION

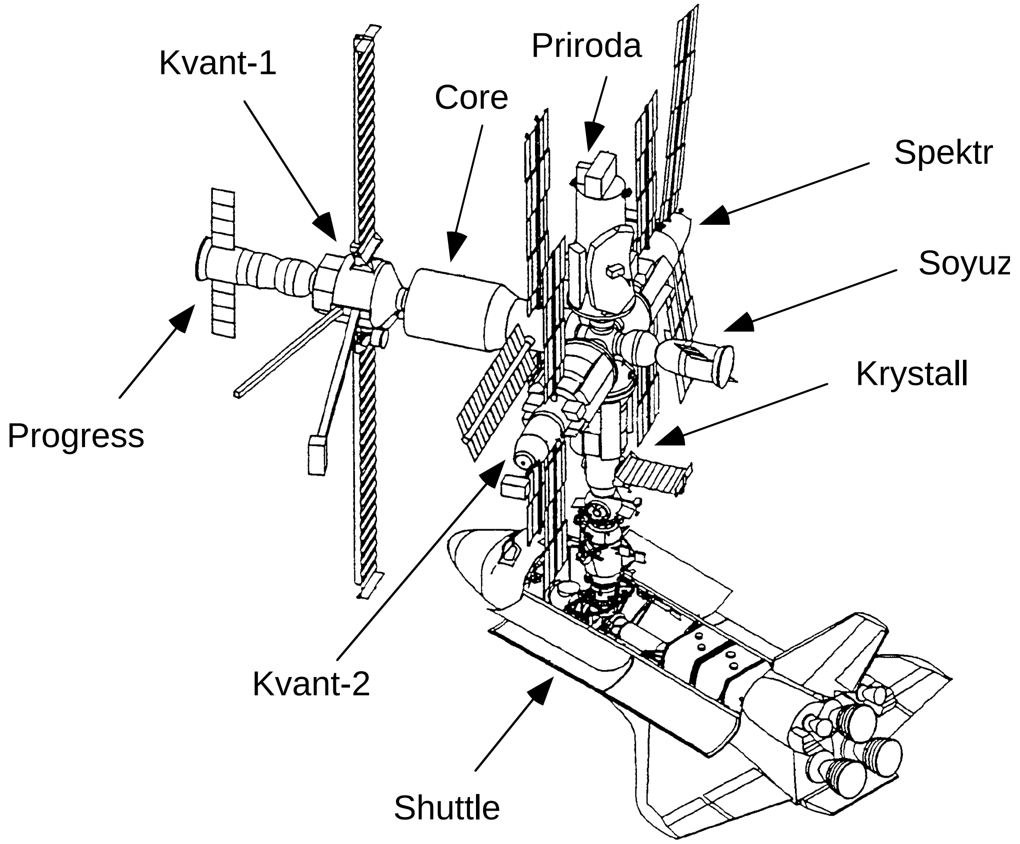


Exact ••••••

Predicted ——

$$\alpha = 0.00005$$

MIR SYSTEM IDENTIFICATION



Identify MIR Markov parameters from response due to Shuttle docking.

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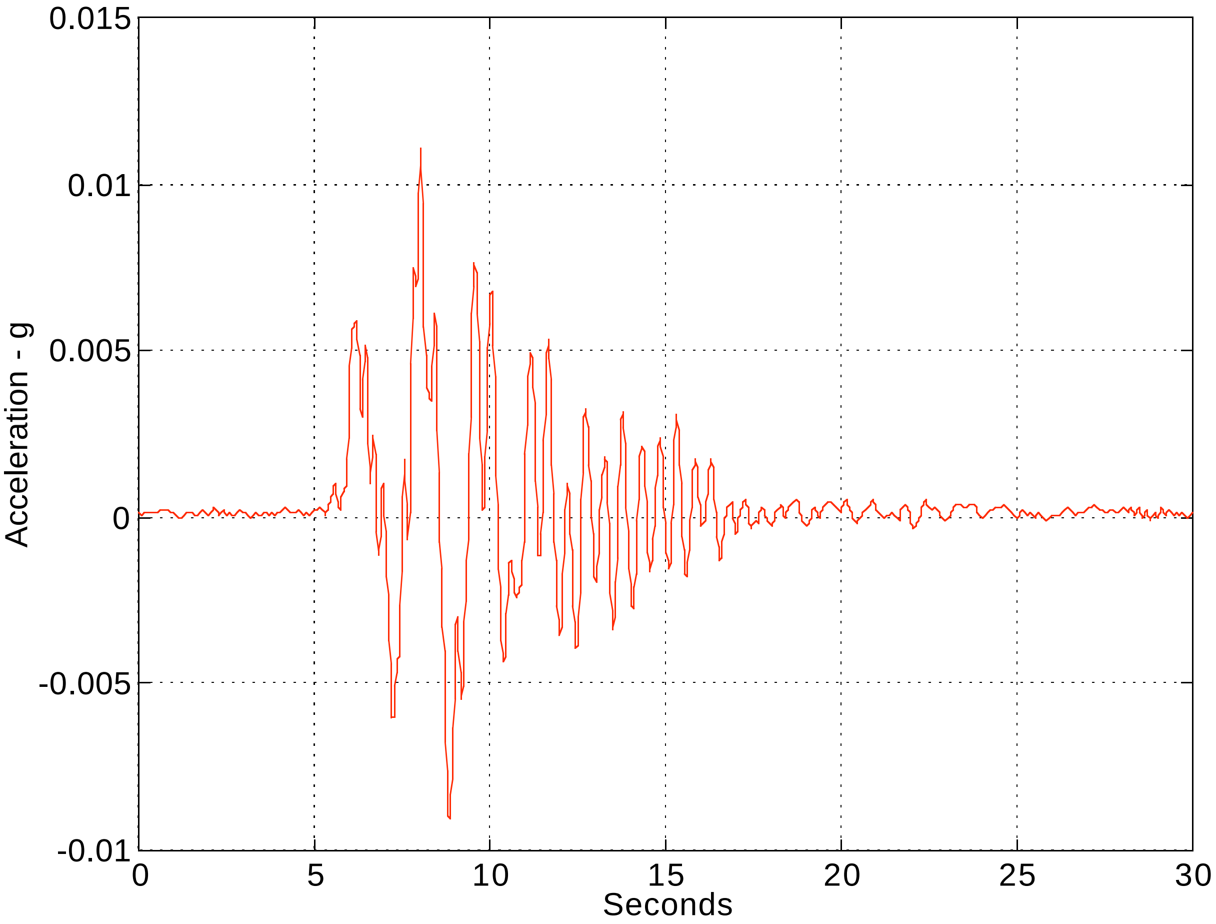
ACCELERATION MEASURED ON KRISTALL - STS-81



Response data filtered to 5.0 Hz. by NASA.

SOLVE FOR MARKOV PARAMETERS

$[H][Y_r]=[Y_e]$

$[H]$ has 1,950 rows and 3,300 columns

$$[H]\begin{bmatrix} [y_{r0}] & [y_{r1}] & \cdots & \cdots & [y_{rn_t-1}] \\ 0 & [y_{r0}] & \cdots & \cdots & [y_{rn_t-2}] \\ 0 & 0 & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & [y_{r0}] & \cdots & [y_{rn_t-N_R}] \end{bmatrix}=\begin{bmatrix}[y_{e0}] & [y_{e1}] & \cdots & [y_{en_t-1}]\end{bmatrix}$$

Compute Moore-Penrose Inverse of $[Y_r]$ using SVD.

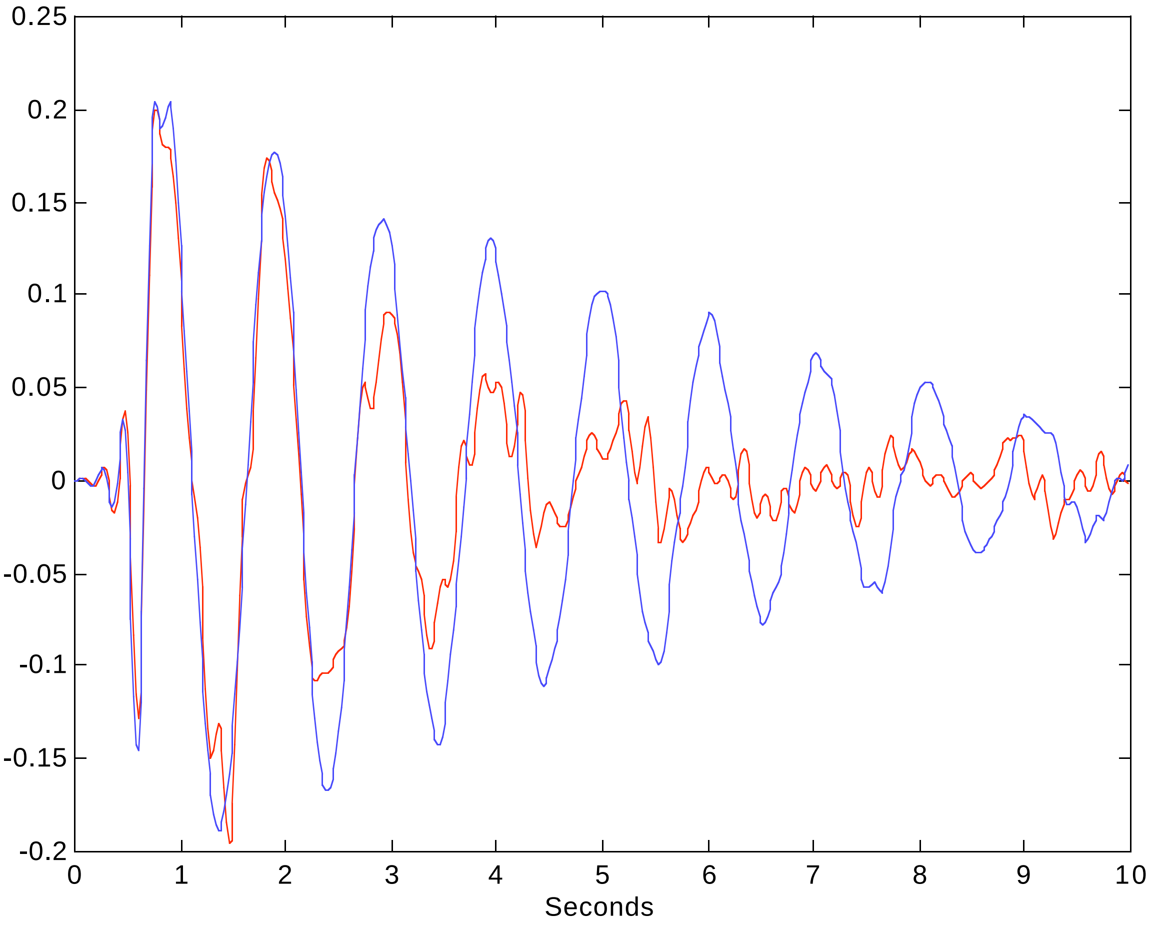
Due to ill-conditioning, SVD algorithm FAILS TO CONVERGE.

Regularize data:

Adjust SVD singular value tolerance.

Add a little artificial noise.

ADD 1.0% RMS NOISE, COMPUTE PSEUDO-INVERSE



Compute Markov parameters and filter to 5.0 Hz.

— **FEM**

— **Docking Data**

SUMMARY

We have only scratched the surface of what there is to know about matrix algebra.

As an experimental or analytical structural dynamicist, you CANNOT do your work without using matrix analysis.

There is a veritable galaxy of neat applications, many of which have not been thought of yet.