

## Biomechanics BME 315

### Notes on waves

Consider the following simple one dimensional problem of waves in a thin rod of linearly viscoelastic material. Waves are transmitted down a long uniform rod of length  $L$ , cross sectional area  $A$ , and density  $\rho$ . The dynamic modulus for a viscoelastic material is complex:  $E^* = E' + i E'' = E'(1 + i \tan \delta)$ , with  $\delta$  as the phase angle between stress and strain. Physically,  $\delta$ , called the 'loss angle', is a measure of energy dissipation in the material. In a one dimensional analysis neglect of Poisson effects is warrantable if the wavelength of stress waves in the rod is much greater than the rod diameter. That is usually not the case for ultrasonic tests. The stress strain relation is taken as

$$\sigma = E^* \epsilon = E^* \frac{u}{z} \quad (3.7.1)$$

in which  $z$  is a coordinate along the rod axis and  $u$  is a displacement in that direction. Consider now a *differential element* of length  $dz$  along the rod. Applying Newton's second law,

$$A dz \frac{\partial^2 u}{\partial t^2} = A \sigma(z+dz, t) - A \sigma(z, t) \quad (3.7.2)$$

Dividing by  $dz$ ,

$$A \frac{\partial^2 u}{\partial t^2} = \frac{A}{z} \sigma \quad (3.7.3)$$

Incorporating the stress strain relation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{E^*}{z^2} \frac{\partial^2 u}{\partial z^2} \quad (3.7.4)$$

This is a *wave equation*. Consider a trial solution of the following form and substitute it. If you prefer,  $\sin(kx - \omega t)$ .

$$u(z, t) = u_0 e^{i(kz - \omega t)}, \quad (3.7.5)$$

$$\frac{\partial^2 u_0}{\partial z^2} = k^2 u_0 \frac{E^*}{z^2} \quad (3.7.6)$$

$$k \sqrt{\frac{E'}{z^2} (1 + i \tan \delta)} = \frac{\omega}{z} \quad (3.7.7)$$

$$k \sqrt{\frac{E'}{z^2} (1 + i \tan \delta)} = \frac{\omega}{z} \quad (3.7.8)$$

Suppose the loss  $\delta$  is small. Expand the second square root and retain the lowest order terms. Moreover, define

$$c = \sqrt{\frac{E'}{\rho}} \quad (3.7.9)$$

$$k = \frac{\omega}{c} (1 + \frac{1}{2} i \tan \delta) \quad (3.7.10)$$

Then Eq. 3.7.3 has a solution,

$$u(z, t) = u_0 \exp\{i (\frac{z}{c} - t)\} \exp\{-\frac{z}{2c} \tan \delta\}. \quad (3.7.11)$$

In the first complex exponential, the argument  $\{z/c - t\}$  is constant if  $dz/c = dt$ , or  $dz/dt = c$ . So, the wave speed  $c$  is interpreted as the *phase velocity* of the wave. As time  $t$  increases,  $z$  must increase to keep the phase argument constant, so the wave moves to the right (direction of increasing  $z$ ). A left moving wave would be written  $u(z, t) = u_0 \exp\{i \{z/c + t\}\}$ .

As for the interpretation of  $k$  (as a real number, for zero loss  $\delta$ ), observe that a sinusoid goes through a complete cycle when its argument goes through  $2\pi$ . So  $kz = 2\pi$  corresponds to one spatial cycle in  $z$ , or one *wavelength*, given the name  $\lambda$ . So  $k = 2\pi / \lambda$ . Similarly, at a given point in space, one cycle occurs when  $\omega t = 2\pi$ . The time for one cycle is called the *period*  $T$  of the wave, and the inverse of the period is the number of cycles per unit time, the *frequency*  $f = 1/T$ . So  $\omega = 2\pi f$ .

The second exponential in Eq. 3.7.11 is a real exponential, and signifies a decrease of wave amplitude with distance  $z$ . This decrease is known as *attenuation*. The attenuation coefficient (per unit length) is defined for small loss as

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| $\frac{\tan \delta}{2c} \quad (3.7.12)$ |
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$\frac{\tan \delta}{2c}$ , the attenuation per unit length, is commonly given in units of nepers per length, in which 'neper' is dimensionless. One neper is a decrease in amplitude of a factor of  $1/e$ . This corresponds to  $z = 1$  in Eq. 3.7.11 with the second term taken as  $e^{-z}$ . The word 'neper' is a degradation of the name of the inventor of the natural logarithm, Napier. The attenuation can also be expressed in decibels (dB) per unit length by the following.  $20 \log_{10}(e) = 8.686$ , so  $(\text{dB/cm}) = 8.68 (\text{neper/cm})$ .

If the rod is elastic ( $\delta = 0$ ), the stress waves propagate without dispersion, that is, the wave speed is independent of frequency, provided the one dimensional assumption is justified by the wavelength being much larger than the rod diameter. Moreover the waves propagate without attenuation. If the rod is viscoelastic, wave speed depends on frequency since  $E'$  depends on frequency, and there is attenuation.