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ORIGINAL CONTRIBUTION

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Interrelation of creep and relaxation for nonlinearly viscoelastic materials: application to ligament and metal

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Abstract Creep and stress relaxation are known to be interrelated in linearly viscoelastic materials by an exact analytical expression. In this article, analytical interrelations are derived for nonlinearly viscoelastic materials which obey a single integral nonlinear superposition constitutive equation. The kernel is not assumed to be separable as a product of strain and time dependent parts. Superposition is fully taken into account within the single integral formulation used. Specific formulations based on power law time dependence and truncated expansions are developed. These are appropriate for weak stress and strain dependence. The interrelated constitutive formulation is applied to ligaments, in which stiffness increases with strain, stress relaxation proceeds faster than creep, and rate of creep is a function of stress and rate of relaxation is a function of strain. An interrelation was also constructed for a commercial die-cast aluminum alloy currently used in small engine applications.

Keywords Creep · Ligaments Metals · Relaxation · Viscoelastic materials

Introduction

Constitutive equations

In linearly viscoelastic materials, the constitutive equation relating time dependent stress $\sigma(t)$ and time dependent strain $\varepsilon(t)$, is a Boltzmann integral:

$$\sigma(t)=\int_0^t E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau,\quad \varepsilon(t)=\int_0^t J(t-\tau)\frac{d\sigma(\tau)}{d\tau}d\tau. \tag{1}$$

where J(t) is the creep compliance as it depends on time t, and E(t) is the relaxation modulus.

Many constitutive models have been developed to describe the nonlinear viscoelastic behavior of materials; see Schapery (1969) and Findley et al. (1976). Models described by Johnson et al. (1996) are for large deformations and Pioletti et al. (1998) take into account the strain rate. In the present study we consider a single-integral form called nonlinear superposition. This allows the relaxation function to depend on strain level and creep to depend on stress:

$$\sigma(t)=\int_0^t E(t-\tau,\varepsilon(\tau))\frac{d\varepsilon(\tau)}{d\tau}d\tau,$$

$$\varepsilon(t)=\int_0^t J(t-\tau,\sigma(\tau))\frac{d\sigma(\tau)}{d\tau}d\tau. \tag{2}$$

A particular form of this, due to Fung (1972), assumes the strain-dependent modulus is *separable* into the product of a function of time and a function of strain:

$$E(t,\varepsilon)=E_t(t)g(\varepsilon). \tag{3}$$

This form has been used widely in the modeling of soft biological tissues, which are nonlinearly viscoelastic. In view of the separable nature of this formulation, it has been referred to as quasi-linear viscoelasticity (QLV). Since in QLV, the time-dependence is separated from the strain-dependence, all the relaxation curves must have the same shape, in general, or the same slope if they exhibit a power-law response. In the more general single-integral nonlinear formulation used here, the shape or slope of the relaxation curves can depend on strain level. It has been shown by Kwon and Kwang (2001), in the case of fluids, that both the differential and integral constitutive equations based on time-strain separability have either Hadamard or dissipative type instabilities. Hadamard instabilities are associated with rapid elastic response. Dissipative instabilities are related to the dissipative viscous nature of the constitutive equations. Both these conditions relate the quality of rheological equations to the laws of thermodynamics. Therefore, the separability hypothesis is invalid for short time periods and is not free of mathematical instabilities.

Multiple integral nonlinear formulations allow for interaction terms when the load or deformation history contains multiple steps (see Findley et al. 1976). For example, a creep and recovery experiment contains two steps.

Interrelation of creep and relaxation

The relationship between creep J(t) and relaxation E(t) in linearly viscoelastic materials is readily obtained via Laplace transformation (see Gross 1968):

$$\int_0^t J(t-\tau)E(\tau)d\tau = \int_0^t E(t-\tau)J(\tau)d\tau = t. \tag{4}$$

Explicit relationships can be obtained from this implicit form via Laplace transformation provided an explicit form is given for E(t) or J(t). Power law behavior in time is particularly simple:

$$E(t) = At^{-n}, \tag{5}$$

with n and A as constants.

The corresponding creep function, with Γ as the gamma function, is

$$J(t) = \frac{1}{A\Gamma(1-n)\Gamma(n+1)}t^n. \tag{6}$$

Equivalently,

$$E(t) = \frac{\sin n\pi}{n\pi}\frac{1}{J(t)}. \tag{7}$$

For a relaxation function consisting of a single exponential plus a constant, the corresponding creep function also contains an exponential, but the creep time constant is longer than the relaxation time constant.

Creep and relaxation are both aspects of the time sensitive behavior of materials, therefore it should be possible to predict one from the other for nonlinear materials as well as for linearly viscoelastic materials considered above. The rationale for seeking an interrelation between creep and relaxation is as follows. Relaxation tests in which a constant strain must be maintained are more difficult to perform than creep tests where stress is kept constant. Creep tests can be performed with a simple dead weight system. By contrast, relaxation tests usually make use of a more complex and costly servo-controlled system. Thus, it would be desirable to perform creep tests and predict the relaxation behavior through a constitutive model. However, interrelation of creep and relaxation for nonlinear materials is not as straightforward as in the linear case.

Many suggested interrelations do not involve superposition. Although they cannot be applied to the general load histories that occur in the body, several are discussed here for completeness. For example, the interrelation of Ashby and Jones (1980) assumes secondary creep, $d\varepsilon/dt = B\sigma^n$, and that the total strain ε is regarded as the sum of an elastic part $\varepsilon_{elastic} = \sigma/E$ and a creep part ε_{creep}. Observe that this does not allow for primary creep or for any linearly viscoelasticity at small strain. The creep is nonlinear at the outset, with no linear term. Ashby and Jones differentiate this and substitute, then integrate from an from an initial stress σ_0 at time $t = 0$ to a final stress σ_f at time t, to obtain the stress relaxation

$$\sigma_f(t) = \left\{BE(n-1)t + \left(1/\sigma_0^{n-1}\right)\right\}^{-1/(n-1)}.$$

The resulting stress relaxation is nearly constant at short time. For $n = 2$, a quadratic nonlinearity, relaxation goes as 1/t at sufficiently long time. Other methods which ignore primary creep were reviewed by Popov (1947).

The interrelation developed by Popov (1947) assumes creep strain as a separable function of time and stress, given as

$$\varepsilon_p = \frac{s_1}{E}\left(e^{\sigma/s_1} - 1\right)T$$

where $T = Ct^m$ is a function of time t and s_1 is a constant. This is a specific type of nonlinearity and time dependence.

The differentiated equation of their relaxation function was

$$\frac{d\varepsilon_e}{dt} + \frac{d\varepsilon_p}{dt} = 0$$

where ε_p is the plastic strain and ε_e is the elastic strain, with

$$\frac{d\varepsilon_p}{dt} = -\frac{1}{E}\frac{d\sigma}{dt}.$$

The creep strain is differentiated and equated to the above form which gives

$$d\sigma = -s_1 \frac{e^{\sigma/s_1} - 1}{1 + Te^{\sigma/s_1}} dT.$$

This is to be arithmetically integrated from zero time and initial stress to give the relaxation function. The final result of the relaxation function which should be obtained by integration is not given explicitly. Also there is no experimental support for the theory.

The Ashby and Popov analyses appear simple in view of the fact that the interrelation between creep and relaxation for linear materials (which are simpler than nonlinear ones) involves a convolution integral as seen in Eq. (4). The simplifying assumption is that the same relation between stress and creep strain rate is valid both under conditions of constant stress (specifically, secondary creep) and constant strain (relaxation). The material may not in fact behave this way. Indeed, the Ashby analysis for secondary creep does not include linear viscoelasticity at small load or primary creep at intermediate load. As for the Popov analysis, Lakes and Vanderby (1999) showed that a separable form for creep leads to a non-separable form for relaxation.

Arutyunyan (1966) wrote non-linear creep in the form of a non-linear differential equation, with the prime denoting a time derivative

$$\sigma'(t) + a\sigma(t) + b\sigma^2(t) = E_0[\varepsilon'(t) + \gamma\varepsilon(t)]$$

This is a rather specific nonlinearity.

After separating the variables and integrating from an initial stress $\sigma_x(\tau_1)$ at time τ_1 to a final stress $\sigma_x(t)$ at time t,

$$\sigma_x(t) = \xi_1 \frac{1 - \alpha_1 e^{-b(\xi_1-\xi_2)(t-\tau_1)}}{1 - \alpha_2 e^{-b(\xi_1-\xi_2)(t-\tau_1)}},$$

where $\alpha_1 = \frac{\xi_2}{\xi_1}\alpha_2$ and $\alpha_2 = \frac{\sigma_x(\tau_1)-\xi_1}{\sigma_x(\tau_1)-\xi_2}$. Since $b(\xi_1 - \xi_2) > 0$, $\sigma_x(t)$ will decrease with time.

The method used above is very complex and there is no use of superposition in the formulation. Also there is no experimental support to the theory.

Touati and Cederbaum (1997) presented a complex and laborious numerical method to convert the Schapery creep model into a set of first order nonlinear equations to predict relaxation. The Schapery model is a single integral constitutive equation with a separable kernel. Again, Lakes and Vanderby (1999) showed that a separable form for creep leads to a non-separable form for relaxation. Guth et al. (1946) emphasizes stress strain curves for rubber at different temperature in which a separable integral equation is briefly mentioned. The inversion proposed by them is not based on superposition, is not supported by analysis, and it does not even properly reduce to linear viscoelasticity as a special case of nonlinearity.

Findley and Lai (1968) have used a multiple integral formulation to interrelate creep and relaxation. Their method utilizes as a first approximation an inversion of a function obtained from multiple integral equation describing creep at constant stress. It is an extension of the linear superposition principle by addition of nonlinear effects in the form of multiple integrals. It is intended for weakly nonlinear materials and does not incorporate explicit superposition in a Stieltjes integral. A product form of kernel functions with time dependence $A + Bt^n + Ct^{2n}$ was used. Parameters in the kernel functions were determined experimentally for a polymer. A cubic function was used to model the nonlinearity,

$$\varepsilon(t) = K_1(t)\sigma + K_2(t)\sigma^2 + K_3(t)\sigma^3.$$

The initial analysis is by the inversion of a power series as a first approximation. Specifically, inversion of the above series to yield σ as functions of ε and K giving rise to the following form:

$$\sigma(t) = f_1(t)\varepsilon + f_2(t)\varepsilon^2 + f_3(t)\varepsilon^3 \text{ where}$$
$$f_1(t) = 1/K_1(t)$$
$$f_2(t) = -K_2(t)/[K_1(t)]^3.$$
$$f_3(t) = \left[2K_2^2(t) - K_1(t)K_3(t)\right]/[K_1(t)]^5.$$

This was inserted in the product form of kernel functions to obtain a new value for strain. Deviation between the new value and constant value served as an input to a numerical approach obtain a second approximation of the stress function. Ordinarily the multiple integral constitutive equations are used to handle stress histories containing multiple steps. For creep or relaxation there is only one step, so a simpler formulation would suffice.

In contrast to the above, we develop herein interrelations based on explicit analysis of superposition within a single integral (non-separable) nonlinear superposition integral constitutive equation.

Ligaments

Ligaments are soft connective tissues composed of closely packed, parallel collagen fiber bundles that are oriented to provide motion and stability (Kastelic et al. 1978). They exhibit time and history dependent stress-strain behavior that is characteristic of viscoelastic materials; see Johnson et al. (1996). Viscoelasticity and nonlinearities in the tissue's mechanical response are important factors in its physiological functions. Creep of ligament is important since load histories in the body contain a static component. Relaxation is relevant to certain stretching exercises for athletic activity.

It has recently been shown that quasi-linear viscoelasticity is insufficient to account for nonlinear material behavior in ligament. In the quasi-linear viscoelasticity (QLV) model of Fung, which is a commonly used phenomenological model of ligament viscoelastic behavior, the relaxation function is separable into the product of a function of time and a function of strain as shown in Eq. (3) above. In Fung's model, the stress is clearly dependent on strain level, but strain does not affect its time dependence. In a graphical form, a purely strain-dependent elastic nonlinearity in QLV manifests itself in the overall height of the relaxation curve and time-dependence manifests itself in the shape of the curve.

Thornton et al. (1997) observed that relaxation in ligament proceeds more rapidly (a greater slope on a log log scale) than creep, a fact not explained by linear viscoelasticity. Lakes and Vanderby (1999) demonstrated via continuum concepts of nonlinear viscoelasticity that such a difference in rate between creep and relaxation occurs when the nonlinearity is of a strain-stiffening type, i.e., the stress-strain curve is concave up as observed in ligament. This was done by developing an interrelation between creep and relaxation, assuming a separable (QLV) model for the creep. The separable form of creep

$$J(t,\sigma) = \{g_1 + g_2\sigma + g_3\sigma^2 + \ldots\}t^n, \tag{8}$$

gives rise to a non-separable relaxation function

$$E(t,\varepsilon) = \{f_1 t^{-n} + f_2 \varepsilon t^{-2n} + f_3 \varepsilon^2 t^{-3n} + \ldots\}(n\pi/\sin\ n\pi). \tag{9}$$

Here, $f_1 = (1/g_1)$ as in the linear case, and $f_2 = -(f_1 g_2/g_1^2)$. Therefore even if a material were found which obeys QLV in creep, it would not obey QLV in relaxation. As for ligament, experiments by Provenzano et al. (2001) showed creep rates (slope on a log-log plot of primary creep) to depend on load level, therefore QLV does not apply to ligament. Pioletti and Rakotomanana (2000) considered the QLV time-strain separability hypothesis but they did not deal directly with the viscoelastic functions. Figures of stress relaxation in Pioletti and Rakotomanana (2000) plotted clearly show that there is a strain dependence so a separable time-strain form as in the QLV model is only an approximation to the actual behavior.

One causal mechanism for the nonlinearity observed in ligament is its fibril arrangement. Under unloaded conditions the microstructure has a wavy appearance, also known as crimp (see Viidik 1968). Collagen fibrils are arranged in varying degrees of crimp such that increasing tensile deformation results in recruitment of additional load bearing fibrils to resist tensile stress. The stress-strain curves of ligament display a concave-up "toe" region where fibers straighten and elongate in a strain-stiffening fashion until they are no longer crimped. Differences in the rates between creep and relaxation are due to this strain-stiffening type of nonlinearity (Lakes and Vanderby 1999).

A number of major factors which can affect the biomechanical and biochemical properties of a ligament include temperature, hydration, aging, immobilization, exercise, irradiation. The movement of fluid within the tissues as well as into and out of the tissues are important factors in the transient behavior of ligaments.

The objective of this paper is to develop an analytical interrelation between creep and relaxation for nonlinearly viscoelastic materials such as ligament. The interrelation is based on a single-integral, nonlinear superposition, viscoelastic model with a non-separable form for relaxation. Several formulations of the model are developed as well as validity analysis for each formulation. The interrelation is applied to experimental data for ligament and metal alloy.

Analysis

Method of interrelation in nonlinear superposition

The single-integral constitutive equation used here is nonlinear superposition. It allows the relaxation function to depend on strain level. Unlike QLV, the slope or the shape of the relaxation curve, not just its magnitude, can depend on strain:

$$\sigma(t) = \int_0^t E(t-\tau, \varepsilon(\tau)) \frac{d\varepsilon}{d\tau} d\tau. \tag{10}$$

Time-dependent strain due to constant creep stress can be written as a sum of immediate and delayed Heaviside step functions in time H(t):

$$\varepsilon(t) = \varepsilon(0)H(t) + \sum_{i=0}^{N} \Delta\varepsilon_i H(t - t_i). \tag{11}$$

This decomposition of creep is illustrated in the top diagram of Fig. 1. Each step strain in the summation gives rise to a relaxing component of stress in view of the definition of the relaxation function. Nonlinearity is accommodated in this analysis since the relaxation function E explicitly depends on strain level:

$$\sigma = \varepsilon(0)E(t,\varepsilon(0)) + \sum_{i=0}^{N} \Delta\varepsilon_i E(t - t_i, \varepsilon(t_i)). \tag{12}$$

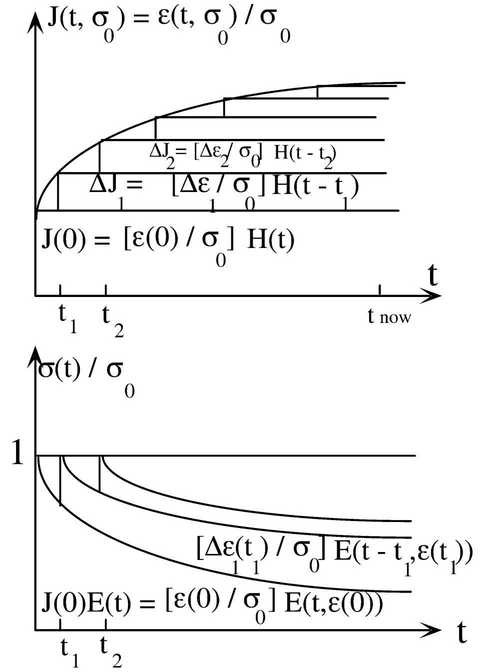

Fig. 1 *Top*: decomposition of a creep function J(t, σ_0) as a sum of immediate H(t) and delayed Heaviside step functions H(t−t_i) in time t. *Bottom*: the constant stress σ_0' which is the same as σ' in the text gives rise to creep expressed as a sum of relaxing components, each of which comes from a step function in the decomposition of the creep curve above

This is shown in the bottom diagram of Fig. 1. Here we assume there is no effect from interactions between the step components; hence we consider single-integral type nonlinear response (Lakes and Vanderby 1999) and exclude response which must be describable by a multiple integral formulation.

Dividing by σ and using the definition of creep compliance,

$$1 = J(0,\sigma)E(t,\varepsilon(0)) + \sum_{i=0}^{N} \Delta J_i E(t - t_i, \varepsilon(t_i)). \tag{13}$$

Pass to the limit of infinitely many fine step components to obtain a Stieltjes integral, with τ as a time variable of integration:

$$1 = J(0,\sigma)E(t,\varepsilon(0)) + \int_0^t E(t-\tau,\varepsilon(\tau))\frac{\partial J(\tau,\sigma)}{\partial\tau}d\tau. \tag{14}$$

The creep compliance J is a function of time and stress. As in the linear interrelation, the time dependence appears in the integral as dependence on a time variable of integration. Since for creep under constant stress, $\sigma(t)=0$ for $t<0$ and $\sigma(t)=\sigma$ for $t>0$, we have $\varepsilon(t)=\sigma J(t,\sigma)$, so Eq. (14) becomes

$$\begin{aligned}1 = {} & J(0,\sigma)E(t,\sigma J(0,\sigma)) \\ & + \int_0^t E(t-\tau,\sigma)J(\tau,\sigma))\frac{\partial J(\tau,\sigma)}{\partial\tau}d\tau.\frac{1}{2}\end{aligned} \tag{14.1}$$

For the linear case this implicit relationship is equivalent to Eq. (4) as can be shown by Laplace transformation. To develop an explicit relationship between creep and relaxation, one assumes a particular functional form for one of the viscoelastic functions. For example, Lakes and Vanderby (1999) used this Stieltjes integral to show that a separable form of creep gives rise to a non-separable relaxation function as described above.

Equation (14) can also be derived in the following way as was suggested by the reviewer. Superposition is still assumed in this approach since it is embodied in the superposition integral used as a starting point:

$$\sigma(t) = \int_0^t E(t-\tau,\varepsilon(\tau))\frac{d\varepsilon(\tau)}{d\tau}d\tau. \tag{14.2}$$

For creep under constant stress, $\sigma(t)=0$ for $t<0$ and $\sigma(t)=\sigma$ for $t>0$ we have

$$\varepsilon(t) = \sigma J(t,\sigma) \tag{14.3}$$

Integration by parts of Eq. (14.2) leads to

$$\sigma(t) = E(0,\varepsilon(t)).\varepsilon(t) - \int_0^t \frac{\partial E(t-\tau,\varepsilon(\tau))}{\partial\tau}.\varepsilon(\tau)d\tau \tag{14.4}$$

Inserting Eq. (14.3) into Eq. (14.4) gives

$$\begin{aligned}\sigma(t) = {} & E(0,\sigma J(t,\sigma)).\sigma J(t,\sigma) \\ & - \int_0^t \frac{\partial E(t-\tau,\sigma J(\tau,\sigma))}{\partial\tau}\sigma J(\tau,\sigma)d\tau\end{aligned} \tag{14.5}$$

Dividing Eq. (14.5) by σ leads to Eq. (14.6) which is the same as Eq. (14.1):

$$1 = J(0,\sigma)E(t,\sigma J(0,\sigma)) + \int_0^t E(t-\tau,\sigma)J(\tau,\sigma))\frac{\partial J(\tau,\sigma)}{\partial\tau}d\tau. \tag{14.6}$$

Thus far, the analysis is exact within the nonlinear superposition constitutive form assumed. To obtain explicit interrelations, several explicit time dependent functions are assumed. Approximations are made at this point since expansions are truncated to low order terms. In the following, various non-separable creep functions involving power laws in time are considered for primary creep. Power laws are used since they are suitable for modeling the behavior of materials of interest. Power law terms have the limitation that the modulus tends to infinity as time tends to zero, an unrealistic situation. Since experimental data used for comparison are available over a limited window of the time domain, this asymptotic behavior is not obtrusive. A semi-inverse approach is used in the analysis.

Two-term nonlinear formulations

Formulation 1

Assume the creep behavior to be as follows, and restrict the analysis to first order in stress throughout:

$$J(t,\sigma) = g_1 t^n + g_2\sigma\left(t^n + At^{2n}\right). \tag{15}$$

We assume a non-separable power law form of relaxation, given as

$$E(t,\varepsilon) \approx f_1 t^{-n} + f_2\varepsilon(t)\left(t^{-n} + t^{-2n}\right) \tag{16}$$

In the following, it is shown that $f_1 = \sin n\pi / n\pi g_1$ as in the linear case, and $f_2 = -(f_1 g_2/g_1{}^2)$ and A is obtained in terms of a function involving gamma functions of n.

The derivative of the creep function is

$$\frac{\partial J(\tau,\sigma)}{\partial\tau} = ng_1\tau^{n-1} + ng_2\sigma\tau^{n-1} + 2nAg_2\sigma\tau^{2n-1}. \tag{17}$$

Since creep strain is

$$\varepsilon(t) = J(t)\sigma \tag{18}$$

from Eqs. (15) and (18) we get

$$\varepsilon(t) = g_1\sigma t^n + g_2\sigma^2\left(t^n + At^{2n}\right) \tag{19}$$

Since we are conducting a first order analysis in σ, we ignore the σ^2 term in Eq. (19). The two term form in Eq. (15) does not have any σ^2 component. Substitute $g_1\sigma t^n$ in Eq. (16) to obtain

$$E(t,\varepsilon(t)) = f_1 t^{-n} + f_2 g_1\sigma + f_2 g_1\sigma t^{-n} \tag{20}$$

Again, this is first order in stress. A second order formulation retaining the σ^2 term is used to develop a relation between the third coefficient g_3 in creep and f_3 in relaxation as given in Formulation 4 below.

Now, substituting Eqs. (20) and (17) into Eq. (14) and recognizing that the first term in the Stieltjes integral vanishes ($J(0) = 0$),

$$1 = \int_0^t \left\{f_1(t-\tau)^{-n} + f_2 g_1\sigma + f_2 g_1\sigma(t-\tau)^{-n}\right\} \left\{ng_1\tau^{n-1} + ng_2\sigma\tau^{n-1} + 2nAg_2\sigma\tau^{2n-1}\right\}d\tau. \tag{21}$$

Now we equate σ independent terms to 1 and all σ dependent terms to 0 from Eq. (21):

$$1 = f_1 g_1 \int_0^t n(t-\tau)^{-n}\tau^{n-1}d\tau$$

$$1 = f_1 g_1 \frac{1}{\sin n\pi} n\pi. \tag{22}$$

The integral portion has the form $\left(\frac{1}{\sin n\pi} n\pi\right)$ and is shown in Appendix A.

Adding all the σ dependent terms together, and ignoring powers of σ greater than 1,

$$0 = \left\{f_1 g_2 \frac{1}{n\pi}\sin n\pi + f_2 g_1^2 \frac{1}{n\pi}\sin n\pi\right\} + \left\{f_1 g_2 A2n\left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)}\right\} + f_2 g_1^2\right\}t^n \tag{23}$$

The proof of solving the above integral is shown explicitly in Appendix B, resulting in

$$f_1 g_2 + f_2 g_1^2 = 0 \tag{24}$$

and

$$f_1 g_2 A\ 2n\left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)}\right\} + f_2 g_1^2 = 0 \tag{25}$$

For Eq. (25) to be equal to Eq. (24),

$$A = \frac{\Gamma(n+1)}{2n\ \Gamma(-n+1)\Gamma(2n)}. \tag{26}$$

Therefore $f_1 g_2 + f_2 g_1{}^2 = 0$, and

$$f_2 = -\left(\frac{f_1 g_2}{g_1^2}\right) \tag{27}$$

So, given a set of creep curves, prediction of the relaxation behavior is possible by following the interrelation of their coefficients obtained by the above formulation. In this formulation, the nonlinear term gives a steeper slope than the linear term. It is most suitable for materials such as metals (Kraus 1980) in which creep accelerates with an increase in stress. It is not appropriate for materials such as ligament (Provenzano et al. 2001) in which creep is slower at elevated stress.

Formulation 2

Assume

$$J(t,\sigma) = g_1 t^n + g_2\sigma t^{n/2} \tag{28}$$

Using the semi-inverse approach, the power of time in the second term of Eq. (29) must be −5n/2 for reasons which become obvious in formulation 3. Specifically, different powers of time are obtained in Eq. (34) which is the σ dependent part of the Stieltjes integral for any value other than $-5n/2$.

So

$$E(t,\varepsilon) \approx f_1 t^{-n} + f_2 \varepsilon(t) t^{-5n/2} \tag{29}$$

In the following, f_2 is found in terms of f_1 , g_1, g_2 and n. The derivative of the creep function is

$$\frac{\partial J(\tau,\sigma)}{\partial \tau} = n g_1 \tau^{n-1} + \frac{n}{2} g_2 \sigma \tau^{(n/2)-1}. \tag{30}$$

Since

$$\varepsilon(t) = J(t)\sigma$$
$$\text{So } \varepsilon(t) = g_1 \sigma t^n + g_2 \sigma^2 t^{n/2}$$

Again, since this formulation is first order in stress, powers of σ greater than 1 are ignored,

$$E(t,\varepsilon(t)) = f_1 t^{-n} + f_2 g_1 \sigma t^{-3n/2} \tag{31}$$

$J(0)=0$. The first term in the Stieltjes integral vanishes.

Substituting Eqs. (31) and (30) in the Stieltjes integral,

$$1 = \int_0^t \left\{ f_1 (t-\tau)^{-n} + f_2 g_1 \sigma (t-\tau)^{-3n/2} \right\} \left\{ n g_1 \tau^{n-1} + \frac{n}{2} g_2 \sigma \tau^{(n/2)-1} \right\} d\tau. \tag{32}$$

Now we equate 'σ' independent terms to 1 and all 'σ' terms to 0 from Eq. (32):

$$1 = f_1 g_1 \int_0^t n (t-\tau)^{-n} \tau^{n-1} d\tau.$$
$$1 = f_1 g_1 \frac{1}{\sin\, n\pi} n\pi \tag{33}$$

Now we take all the 'σ' terms:

$$0 = f_1 g_2 \frac{n}{2} \left\{ \frac{\Gamma(-n+1)\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{-n}{2}+1\right)} \right\} + f_2 g_1^2 n \left\{ \frac{\Gamma\left(\frac{-3n}{2}+1\right)\Gamma(n)}{\Gamma\left(\frac{-n}{2}+1\right)} \right\} \tag{34}$$

Canceling out the common terms from Eq. (34) we get

$$0 = f_1 g_2 \left\{ \frac{\Gamma(-n+1)\Gamma\left(\frac{n}{2}\right)}{2} \right\} f_2 g_1^2 \left\{ \Gamma\left(\frac{-3n}{2}+1\right)\Gamma(n) \right\}$$

Therefore solving for f_2, we obtain

$$f_2 = \frac{-f_1 g_2 \Gamma(-n+1)\Gamma\left(\frac{n}{2}\right)}{2 g_1^2 \Gamma\left(\frac{-3n}{2}+1\right)\Gamma(n)} \tag{35}$$

The ratio $\frac{\Gamma(-n+1)\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{-3n}{2}+1\right)\Gamma(n)}$ in Eq. (35) is almost equal to 1 so for small values of slope n, the relation reduces to

$$f_2 = \frac{-f_1 g_2}{g_1^2} \tag{36}$$

This formulation gives a very flat curve for the ligament creep data as the value of g_2 obtained is negative and the slope of the second time term is n/2. So in the following formulation, the power of the second time term was kept independent of 'n' providing more flexibility in fitting ligament creep data which correspondingly helps in making a better prediction of relaxation.

Formulation 3

Assume

$$J(t,\sigma) = g_1 t^n + g_2 \sigma t^m \tag{37}$$

Now assume

$$E(t,\varepsilon) \approx f_1 t^{-n} + f_2 \varepsilon(t) t^{-q} \tag{38}$$

in which f_2 and q are to be determined by the analysis. It is shown that $f_1 = (1/g_1)(\sin n\pi / n\pi)$ and that $q = 3n - m$.

The derivative of the creep function is

$$\frac{\partial J(\tau,\sigma)}{\partial \tau} = n g_1 \tau^{n-1} + m g_2 \sigma \tau^{m-1} \tag{39}$$

Since

$$\varepsilon(t) = J(t)\sigma$$
$$\text{So } \varepsilon(t) = g_1 \sigma t^n + g_2 \sigma^2 t^m \tag{40}$$

Again since this formulation is first order in stress, powers of greater than 1 are ignored,

$$E(t,\varepsilon(t)) = f_1 t^{-n} + f_2 g_1 \sigma t^{-q+n}$$
$$J(0) = 0, \tag{41}$$

since the power law form requires the first term in the Stieltjes integral to vanish.

Substituting Eqs. (41) and (39) in the Stieltjes integral (Eq. 14), we get

$$1 = \int_0^t \left\{ f_1 (t-\tau)^{-n} + f_2 g_1 \sigma (t-\tau)^{-q+n} \right\} \left\{ n g_1 \tau^{n-1} + m g_2 \sigma \tau^{m-1} \right\} d\tau. \tag{42}$$

Now we equate 'σ' independent terms to 1 and all 'σ' terms to 0 from Eq. (42):

$$1 = f_1 g_1 \int_0^t n(t-\tau)^{-n}\tau^{n-1} d\tau.$$

$$1 = f_1 g_1 \frac{1}{\sin\ n\pi n\pi}. \tag{43}$$

So the relation between f_1 and g_1 is the same as in the linear case.

Now we take all the 'σ' terms:

$$0 = f_1 g_2 m \left\{ \frac{\Gamma(-n+1)\Gamma(m)}{\Gamma(m-n+1)} \right\} t^{m-n} + f_2 g_1^2 n \left\{ \frac{\Gamma(-q+n+1)\Gamma(n)}{\Gamma(-q+2n+1)} \right\} t^{-q} + 2n \tag{44}$$

For Eq. (44) to be compatible, the power of t has to be the same so

$$\begin{aligned} m - n &= -q + 2n \\ q &= 3n - m. \end{aligned} \tag{45}$$

This is the slope of the predicted relaxation curve.

Substituting Eq. (45) into Eq. (44), we obtain

$$0 = f_1 g_2 m \left\{ \frac{\Gamma(-n+1)\Gamma(m)}{\Gamma(m-n+1)} \right\} t^m - n + f_2 g_1^2 n \left\{ \frac{\Gamma(m-2n+1)\Gamma(n)}{\Gamma(m-n+1)} \right\} t^{m-n} \tag{45}$$

Canceling out the common terms from the above equation and solving for f_2, we obtain

$$f_2 = \frac{-f_1 g_2 m \Gamma(-n+1)\Gamma(m)}{n g_1^2 \Gamma(-2n+m+1)\Gamma(n)}. \tag{46}$$

This gives the amplitude of the predicted relaxation curve.

The procedure for curve fitting nonlinear data to obtain g_1, g_2, n, and m is described later. A strain stiffening nonlinearity such as that seen in ligament forces g_1 to be positive and g_2 to be negative in Eq. (37). Since the predicted relaxation slope is $q = 3n - m$, this model also shows relaxation proceeds more rapidly (a greater slope on a log log scale) than creep as a result of the strain-stiffening nonlinearity provided $3n - m > m$. Results of the interrelation for ligament using the above formulation are shown in Figs. 4 and 5.

Three-term nonlinear formulation

Formulation 4

The above procedure is again followed with a more complex creep function $J(t,\sigma)$. A, B, W, X, and Y are constants:

$$J(t,\sigma) = g_1 t^n + g_2 \sigma \left(t^n + A t^{2n}\right) + g_3 \sigma^2 \left(t^n + A t^{2n} + B t^{3n}\right) \tag{47}$$

$$\frac{\partial J(\tau,\sigma)}{\partial \tau} = n g_1 \tau^{n-1} + n g_2 \sigma \tau^{n-1} + 2nA g_2 \sigma \tau^{2n-1} + n g_3 \sigma^2 \tau^{n-1} + 2nA g_3 \sigma^2 \tau^{2n-1} + 3nB g_3 \sigma^2 \tau^{3n-1} \tag{48}$$

$$E(t,\varepsilon) \approx f_1 t^{-n} + f_2 \varepsilon(t)\left(t^{-n} + X t^{-2n}\right) + f_3 \varepsilon(t)^2 \left(W t^{-n} + X t^{-2n} + Y t^{-3n}\right) \tag{49}$$

Using Eqs. (18) and (47) we get

$$\text{So}\ E(t,\varepsilon) = f_1 t^{-n} + f_2 \left(t^{-n} + X t^{-2n}\right)\left[g_1 \sigma t^n + g_2 \sigma^2 \left(t^n + A t^{2n}\right)\right] + f_3 \left(W t^{-n} + X t^{-2n} + Y t^{-3n}\right)\left[g_1^2 \sigma^2 t^{2n}\right] \tag{50}$$

Substituting Eqs. (48) and (50) in the Stieltjes integral given by Eq. (14) and knowing that J(0) term in it vanishes we get

$$\begin{aligned} 1 = \int_0^t \{ & f_1 (t-\tau)^{-n} + f_2 g_1 \sigma + f_2 g_1 X \sigma (t-\tau)^{-n} + f_2 g_2 \sigma^2 \\ & + f_2 g_2 X \sigma^2 (t-\tau)^{-n} + f_2 g_2 A \sigma^2 (t-\tau)^n + f_2 g_2 A X \sigma^2 \\ & + f_3 g_1 W^2 \sigma^2 (t-\tau)^n + f_3 g_1^2 W X \sigma^2 \\ & + f_3 g_1^2 W Y \sigma (t-\tau)^{-n} \} \{ n g_1 \tau^{n-1} + n g_2 \sigma \tau^{n-1} + 2nA g_2 \sigma \tau^{2n-1} \\ & + n g_3 \sigma^2 \tau^{n-1} + 2nA g_3 \sigma^2 \tau^{2n-1} + 3nB g_3 \sigma^2 \tau^{3n-1} \} d\tau. \end{aligned} \tag{51}$$

Again, 'σ' independent terms are equated to 1 and all 'σ' dependent terms to 0:

$$1 = f_1 g_1 \int_0^t n(t-\tau)^{-n} \tau^{n-1} d\tau.$$

$$1 = f_1 g_1 \frac{1}{\sin\ n\pi} n\pi$$

(52)

Now we take all the 'σ' terms:

$$0 = \left\{ f_1 g_2 \frac{1}{\sin\ n\pi} n\pi + f_2 X g_1^2 \frac{1}{\sin\ n\pi} n\pi \right\} + \left\{ f_1 g_2 . A . 2n \left\{ \frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)} \right\} + f_2 g_1^2 \right\} t^n \tag{53}$$

So from Eq. (53) we get Eqs. (54) and (55)

$$f_1 g_2 \frac{1}{\sin\ n\pi} n\pi + f_2 X g_1^2 \frac{1}{\sin\ n\pi} n\pi = 0 \tag{54}$$

$$f_1 g_2 . A . 2n \left\{ \frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)} \right\} + f_2 g_1^2 = 0 \tag{55}$$

Now we take all the 'σ^2' terms:

$$
\begin{aligned}
0= & \left\{f_1 g_3 \frac{1}{\sin n\pi} n\pi + 2 f_2 g_1 g_2 X \frac{1}{\sin n\pi} n\pi + W f_3 Y g_1^3 \frac{1}{\sin n\pi} n\pi\right\} \\
& + \left\{ f_1 g_3 A 2n \left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)}\right\}\right. \\
& \left. + \left[2+A+AX2n\left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)}\right\}\right] f_2 g_1 g_2 + W f_3 X g_1^3\right\} t^n \\
& + \left\{ f_1 g_3 B 3n \left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(2n+1)}\right\}\right. \\
& + f_2 g_1 g_2 \left[A + An\left\{\frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n+1)}\right\}\right] \\
& \left. + W f_3 g_1^3 n \left\{\frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n+1)}\right\}\right\} t^{2n}
\end{aligned} \tag{56}
$$

From Eq. (56) we get Eqs. (57), (58), and (59):

$$
\begin{aligned}
& f_1 g_3 \frac{1}{\sin\ n\pi} n\pi + 2 f_2 g_1 g_2 X \frac{1}{\sin\ n\pi} n\pi \\
& + W f_3 Y g_1^3 \frac{1}{\sin\ n\pi} n\pi = 0
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
& f_1 g_3 A 2n \left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)}\right\} \\
& + \left[2+A+AX2n\left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n+1)}\right\}\right] f_2 g_1 g_2 + W f_3 X g_1^3 = 0
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
& f_1 g_3 B 3n \left\{\frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(2n+1)}\right\} \\
& + f_2 g_1 g_2 \left[A + An\left\{\frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n+1)}\right\}\right] \\
& + W f_3 g_1^3 n \left\{\frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n+1)}\right\} = 0
\end{aligned} \tag{59}
$$

Let

$$
\begin{aligned}
a &= \frac{1}{\sin\ n\pi} n\pi \\
b &= \frac{g_2}{g_1} \\
c &= f_2 g_1^2 \\
d &= \frac{\Gamma(-n+1)\Gamma(2n)}{\Gamma(n)} \\
f &= \frac{g_3}{g_1} \\
g &= f_2 g_1 g_2 \\
h &= f_3 g_1^3 \\
j &= \frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n)} \\
k &= \Gamma(-n+1)
\end{aligned}
$$

The above variables were substituted into Eqs. (54), (55), (57), (58), and (59) and solved for A, B, X, Y, and W. Mathematica (Wolfram Research Inc, 100 Trade Center Drive, Champaign, IL 61820 Version 4.0) was used in solving these equations and the reason the above substitutions were used was to simplify the equations for the software and decrease computation time.

The values obtained are given as follows:

$$
\begin{aligned}
A &= \frac{-ac}{2bd} \\
B &= \frac{a\left(2abcg + 2ac^2 dfj + abcgj + a^2c^2 gj - 6abcdgj\right)}{6b^2 dfk} \\
X &= \frac{-b}{ac} \\
Y &= \frac{2\left(-b^2 cdf + 2b^3 dg\right)}{a^2c^2(2cdf + acg - 6bdg)} \\
W &= \frac{-2ac^2 df - a^2c^2 g + 6abcdg}{2b^2 dh}
\end{aligned}
$$

The forms obtained here for A, B, X, Y, and W are quite complex. Calculation of their values and entering them into Eqs. (47) and (49) would be an arduous task. For the experimental data considered, a two term formulation models creep within a reasonable degree of accuracy. Then, a three term fit is avoided.

Applications of the interrelations

The interrelations (formulation 3) were applied to primary creep data for ligament and for aluminum alloy as follows.

Ligament data were collected for both creep and relaxation at different stress and strain levels respectively (R. Hingorani (unpublished)). Rabbit medial collateral ligament was tested for a period of 100 s followed by a period of recovery which lasted 1000 s. Relaxation tests were run consecutively at different strain levels. The stress relaxation was carried out first and the corresponding creep test was carried out on the contralateral ligament. No preconditioning was done on the ligament, however a preload of 0.5 N was applied to the ligament.

Ligament data were plotted on a log-log scale with the first point plotted at 1 s into the test. The rise time was 0.1 s. It was seen that the results were non-linear in time and stress or strain within the physiologic range which was also pointed out by Provenzano et al. (2001) for rat ligament. The creep rate was seen to decrease with higher levels of stress and the rate of relaxation was seen to decrease at higher levels of strain.

The method used to fit the curves was as follows. The time scale of the creep and relaxation was divided by t_1 (1.5 s for the ligament data and 10 s for the metal data) in order to simplify calculation of powers. Isochronal plots of strain vs. stress were generated for two different times. The first isochronal was curve fitted with $\varepsilon = g_1\sigma + g_2\sigma^2$ based on formulation 3 to obtain the

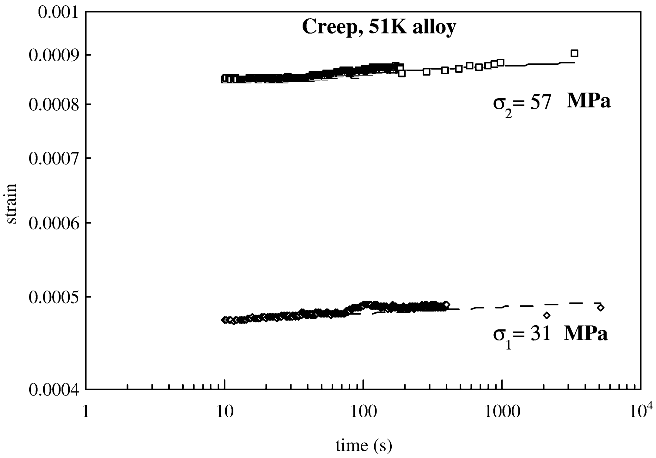

Fig. 2 Curve fitting of creep of aluminum alloy 51 K. Data of aluminum creep at two different stress levels σ_1 (31 MPa) and σ_2 (57 MPa). g_1 and g_2 are obtained from the first isochronal and these values are used in the 2nd isochronal to get n and m. These values are used to fit the creep curve at σ_1 (31 MPa) and σ_2 (57 MPa) with $\varepsilon(t) = g_1\sigma t^n + g_2\sigma t^m$. The *points* give the experimental results while the *dashed line* is the curve fit

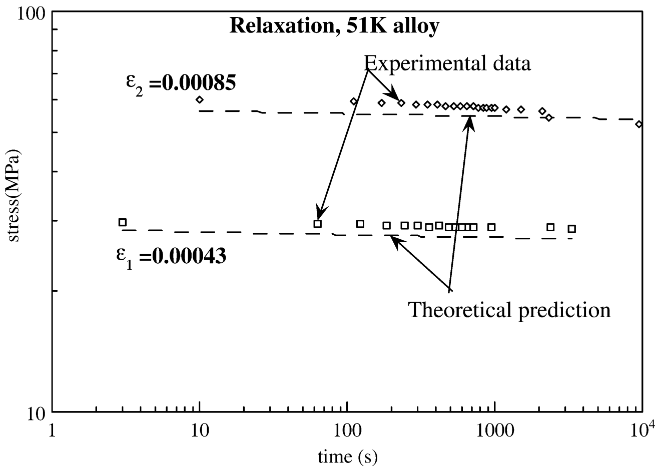

Fig. 3 Prediction of relaxation from creep; comparison with experimental relaxation of alloy. The two corresponding strain levels for relaxation are ε_1 ($430\text{x}10^{-6}$) and ε_2 ($850\text{x}10^{-6}$). Similarly relaxation data of aluminum for two different strain levels are predicted very well by the interrelation used in Formulation 3. The *points* give the experimental results while the *dashed line* is the theoretical prediction

values for g_1 and g_2. The second isochronal at t_2 (90 s for the ligament data and 388 seconds for the metal data) was curve fitted with Eq. (37) with known values of g_1 and g_2 to obtain n and m. These values of g_1, g_2, n, and m were used to fit the different creep curves with Eq. (37). Using the interrelation in Eqs. (43), (45), and (46), the predicted relaxation curve was generated.

A strain stiffening kind of nonlinearity is observed in ligaments which means that for a large change in stress there is a small change in strain. So in the creep compliance equation, $J(t) = g_1 t^n + g_2\sigma t^m$ as the level of σ increases, J(t) has to correspondingly decrease due to the strain stiffening kind of nonlinearity. For this to happen, g_2 has to be less than zero which is observed while fitting the creep curves. Similarly, as the level of strain increases, the relaxation modulus has to increase. So in the relaxation modulus equation, $E(t) = f_1 t^{-n} + f_2\varepsilon t^{(-3n-m}$)as the level of strain increases, E(t) has to correspondingly increase. For this, f_2 has to be greater than zero which is consistent with our results.

An additional comparison was done for creep of aluminum alloy. Data were collected for both creep and relaxation at different stress and strain levels respectively (T. Jaglinski (unpublished)). Aluminum-silicon alloy was tested at 220 °C at 31 MPa and 57 MPa for creep and $430\times10^{-6}$and 850×10^{-6} strain for relaxation. The rise time was 2 s for creep. Data were plotted on a log-log scale with the first point plotted at 10 s into the test. Results of the curve fit of creep for aluminum alloy are shown in Fig. 2, and comparison of predicted and experimental relaxation shown in Fig. 3. Corresponding results for ligament are shown in Figs. 4 and 5. Both the modeling and the prediction of relaxation was of

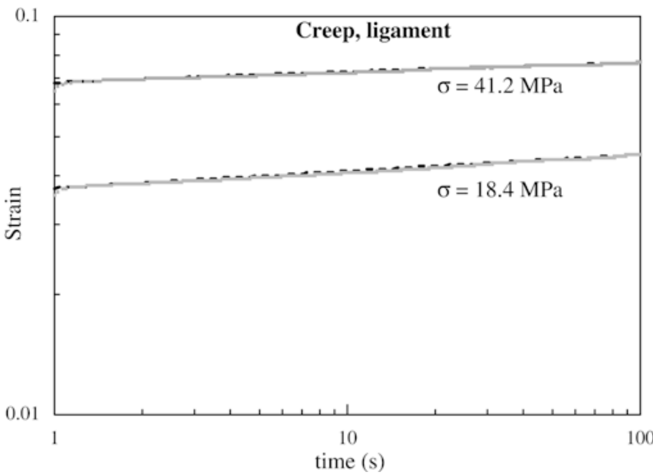

Fig. 4 Curve fitting of creep of ligament. Data of rabbit medial collateral ligament at two different stress levels σ_1 (18.4 MPa) and σ_2 (41.2 MPa). *Points (dense)*: experiment. g_1 and g_2 are obtained from the first isochronal and these values are used in the 2nd isochronal to get n and m. These values are used to fit the creep curve at σ_1 (18.4 MPa) and σ_2 (41.2 MPa) with $\varepsilon(t) = \sigma[g_1 t^n + g_2\sigma t^m]$. *Dash line*: curve fit

good quality over the window of time and strain considered.

The analysis has the following limitations. The model for creep and relaxation that we have used for comparison with experiment is a first order one in stress dependence. Therefore it is appropriate for weak nonlinearity such as is seen in a restricted window of stress and time in the experimental results used for comparison. Within that window it predicts the relaxation very well from the creep.

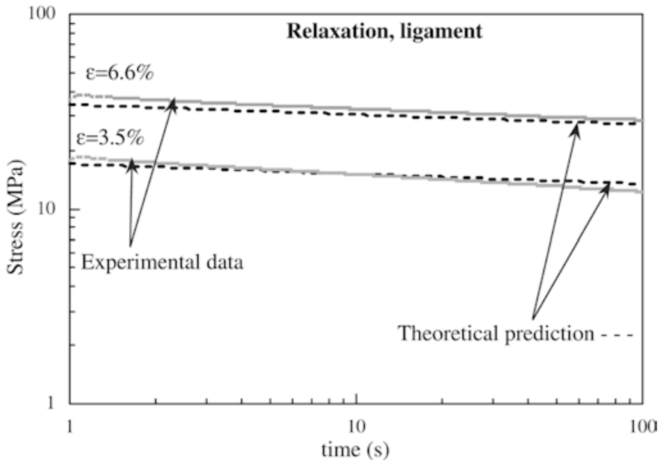

Fig. 5 Prediction of relaxation from creep; comparison with experimental relaxation of ligament. The two corresponding strain levels for relaxation are ε_1 (3.5%) and ε_2 (6.6%). Similarly relaxation data of rabbit medial collateral ligament for two different strain levels are predicted very well by the interrelation used in Formulation 3. The *solid line* in the figure gives the experimental curve while the *dashed line* is the theoretical prediction

The superposition approach developed here is amenable to higher order expansion of the kernel as illustrated in formulation 4 above. Such formulations could provide a wider window of applicability at the cost of additional complexity. It is also possible that the single integral constitutive equation considered here may not suffice regardless of the kernel. In that case, a multiple integral constitutive equation may be used.

Conclusions

In this work, creep and stress relaxation are interrelated for primary creep described by a sum of power-law terms in time, within the framework of single integral nonlinear superposition. If the nonlinearity is of a strain-stiffening type, as is the case with ligament, relaxation proceeds more rapidly (a greater slope on a log log scale) than creep. If the nonlinearity is of a strain-softening type, relaxation proceeds more slowly than creep.

A two-term interrelation applied to stress-dependent creep for ligament gives rise to excellent prediction of nonlinear stress relaxation of ligament.

Appendix A

From the proof given below in Appendix B we get

$$\int_0^t n\tau^{-n}(t-\tau)^{n-1}d\tau = n\,\Gamma(-n+1)\Gamma(n). \tag{A1}$$

However there is a standard identity (Artin 1964) for the product of gamma functions on the right hand side in Eq. (A1) which is given as

$$\Gamma(-n+1)\Gamma(n) = \frac{\pi}{\sin\, n\pi}$$

So Eq. (A1)can therefore be written as

$$\int_0^t n.\tau^{-n}(t-\tau)^{n-1}d\tau = \frac{n\pi}{\sin\, n\pi}. \tag{A2}$$

Appendix B

The method used to solve the integral of type used in the formulation is as follows:

$$\int_0^t \tau^{m-1}(t-\tau)^{n-1}d\tau. \tag{B1}$$

Rewriting Eq. (B1) in a different form we get

$$\int_0^t \tau^{m-1}t^{n-1}\left(1-\frac{\tau}{t}\right)^{n-1}d\tau. \tag{B2}$$

Let $\frac{\tau}{t} = u$

therefore $\tau = ut$

therefore $d\tau = t\,du$

So when $\tau = 0$, $u = 0$and when $\tau = t$, $u = 1$

Substituting the above result in Eq. (B2), we get

$$\begin{aligned}&\int_0^t (ut)^{m-1}t^{n-1}(1-u)^{n-1}t\,du.\\ &t^{m+n-1}\int_0^t u^{m-1}(1-u)^{n-1}\,du.\end{aligned} \tag{B3}$$

The integral part in Eq. (B3) is nothing but a Beta function (Andrews 1985), the definition of which is given as

$$B(m,n) = \int_0^t \tau^{m-1}(1-\tau)^{n-1}d\tau = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

So the final result can be written as

$$t^{m+n-1}B(m,n) = t^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \tag{B4}$$

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