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THERMODYNAMIC EQUIVALENCE OF STEADY-STATE SHOCKS AND SMOOTH WAVES IN GENERAL MEDIA; APPLICATIONS TO ELASTIC-PLASTIC SHOCKS AND DYNAMIC FRACTURE

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ABSTRACT

By comparing the First Law of thermodynamics in its shock wave form to its smooth wave form, and applying standard continuum mechanical conservation laws and geometrical compatibility, we prove *for arbitrary media* that a shock wave which propagates without rotating under steady-state conditions is thermodynamically identical to a suitably-chosen steadily propagating smooth wave (and that this is not so in general for nonsteady shocks). This legitimizes the derivation of restrictions on steady-state shock waves by the analysis of suitably-chosen steady smooth waves in purely mechanical material models. Doing so for a broad class of rate-independent elastic–plastic materials rigorously corroborates several recently-published shock restrictions whose derivations involved some (now validated) heuristic arguments, and substantially generalizes the material class for which these restrictions apply. Thus, e.g. within small-displacement-gradient theory, stress jumps are ruled out across steadily propagating shock waves in quasi-static deformations of any nonsoftening material satisfying plastic normality and positive-definiteness of the elastic modulus tensor (removing the previous limitation of this result to materials that satisfy the global maximum plastic work inequality and whose current yield locus always incorporates all prior yield loci). We also confirm that steady-state shock waves in dynamic anti-plane strain or plane strain deformations cannot exist except at elastic wave speeds for nonhardening materials in the same broad constitutive class unless the yield surface contains a linear segment. Application of these results to steady-state dynamic subsonic plane strain crack growth in elastic–ideally plastic Prandtl–Reuss–Mises material proves that this problem's solution must be shock-free. This implies that certain solutions containing strong discontinuity surfaces, obtained in a recently-published numerical finite element study of this dynamic crack growth problem, are not physically realizable. The conclusion is that either a more robust numerical procedure is necessary which incorporates the thermodynamics-mandated shock restrictions derived here, or that *steady-state* subsonic dynamic plane-strain elastic–plastic crack growth is not possible in this material model (and potentially not in nature for materials exhibiting plastic normality, purely nonlinear yield surfaces and no hardening).

Keywords: A. shock waves, A. dynamic fracture, A. thermomechanical processes, B. elastic–plastic material, B. metallic materials.

1. INTRODUCTION

Because of the great complexity and indeterminacy of form associated with the use of fully thermomechanical constitutive equations, most continuum mechanics boundary value problems are formulated in terms of purely mechanical constitutive equations. While such simplification is essential for tractability in most cases, one still wishes not to exclude the modeling of phenomena that can actually occur in nature, nor to introduce phenomena that are physically impossible. An important example of this is shock waves. In actual materials, the large gradients in material velocity and temperature associated with a shock wave produce marked effects of friction and heat conduction, causing irreversible thermodynamic processes and hence entropy production. In general in such large gradient regions, then, purely mechanical constitutive equations are not sufficient to capture the physical phenomena. However, these regions are usually observed to be very narrow; thus, an alternative to the use of a fully thermomechanical constitutive formulation is to employ purely mechanical constitutive equations, but permit the existence of a sharp (jump) discontinuity in the appropriate field variables to capture the physical phenomenon of a shock wave. When this latter approach is adopted, it is obviously of crucial importance that a careful determination be made of the conditions under which such jump discontinuities should be admitted, and of the types of discontinuities allowed, when solutions are sought to boundary value problems involving purely mechanical constitutive equations. This is crucially important since if only continuous (i.e., smooth) solutions are sought to purely mechanical boundary-value problems, important physical phenomena may be missed, whereas if jump discontinuities are inappropriately permitted in the analysis of such problems, "solutions" may be found that lack physical counterparts.

There have been several analyses addressing this issue of when moving jump discontinuity surfaces should be permitted, and if so what types, in the solution of purely mechanical boundary-value problems, especially for elastic–plastic solids. Until very recently, these analyses have employed a heuristic argument to require that the *jump* forms of the standard continuum mechanical conservation laws and geometrical compatibility should apply in *incremental* forms to the stress and deformation paths through a shock wave, together with incremental forms of purely mechanical constitutive restrictions. Thus, Drugan and Rice (1984) and Drugan (1986) showed that, within a quasi-static and small-displacement-gradient framework, the skeletal constitutive assumptions of the maximum plastic work inequality and positive-definiteness of elastic strain energy density are sufficient to rule out propagating discontinuities of stress (not merely traction) and to restrict severely admissible types of propagating strain and velocity discontinuities, for materials that either are nonhardening or that harden in such a way that the current yield locus always incorporates all prior yield loci, but are otherwise arbitrarily anisotropic. Drugan and Shen (1987) extended this analysis to dynamic deformations of the same class of material response, and showed in particular that for anti-plane strain and incompressible plane strain deformations of nonhardening material, dynamically propagating surfaces of jump discontinuity in stress and/or material velocity are not admissible except at elastic wave speeds or unless the yield surface contains a "flat" which then dictates the admissible propagation speed; Leighton *et al.* (1987) independently proved inadmissibility for the specific case of incompressible plane strain of an isotropic nonhardening Prandtl–Reuss–Mises material. In a fascinating application, Nikolic and Rice (1988) analyzed anti-plane shear dynamic crack propagation in an elastic–ideally plastic single crystal (whose yield surface in stress space *does* contain flats), showing that near-tip solutions necessarily contain propagating jump discontinuities that satisfy Drugan and Shen's (1987) conditions. Further discontinuity work includes Drugan and Shen's (1990) generalization of the previous quasi-static and dynamic analyses to finite deformations, Shen and Drugan's (1990) treatment of the plane stress and *compressible* plane strain cases, and Brannon and Drugan's (1993) exploration of the influence of non-classical constitutive features, such as non-normality of the plastic strain increments to the yield surface, to admissibility of propagating jump discontinuities.

Brannon *et al.* (1995) initiated an effort to explore rigorously the validity of the aforementioned heuristically-argued assumptions, upon which the above-reviewed analyses are based. Brannon *et al.* proved that during weak shock passage, a material particle's stress and deformation history is well-approximated by its history during passage of a smooth wave with (i) purely mechanical constitutive response and (ii) stress and deformation paths that satisfy incremental versions of the jump forms of the conservation laws and geometrical compatibility. The first of these was accomplished by generalizing a one-dimensional inviscid fluid analysis of Courant and Friedrichs (1948) to general three-dimensional deformations of any magnitude in arbitrary materials. The specific approach of Courant and Friedrichs that yields results directly addressing the question at hand, and hence the one that Brannon *et al.* (1995) generalized, is that in which one analyzes the thermodynamic differences between a weak shock and a smooth wave. To do this, one first constructs the Hugoniot function, which is a restatement of the First Law of thermodynamics in the jump form valid across a shock: this function thus provides the set of all admissible values of field variables on one side of a shock given a specific set of these variables on the other side; the function is defined such that it is zero when the First Law is satisfied. To compare a weak shock with a smooth wave, one calculates a Taylor expansion of the Hugoniot function about the leading (or trailing) state of the smooth wave, and uses the smooth forms of the conservation laws to evaluate it. The order at which, and manner in which, the Taylor expansion diverges from zero then reveals the thermodynamic differences between a shock and a smooth wave, which are due to the additional entropy production that in general occurs in a shock as compared to a smooth wave. This is because entropy production by a shock shows up as a modification to the First Law of thermodynamics, as compared to that law's form for smooth waves in a purely mechanical model. Brannon *et al.* (1995) showed that for general deformations in arbitrary media, a smooth wave coincides with a shock up until third order in material time rates of fundamental field variables, at which degree they differ, in general. This conclusion concurs with that arrived at by Courant and Friedrichs (1948) for the special one-dimensional inviscid fluid case they considered. The implication of this result is that a general *weak* shock is well-approximated by a smooth wave in a purely mechanical constitutive model.

Brannon *et al.* (1995) also considered the case of *steady-state* shock propagation. Here they addressed item (ii) above, proving that the incremental requirements of the conservation laws and compatibility through a smooth wave rigorously reduce, when steady-state wave propagation is considered, to precisely the *incremental* forms of the shock (jump) conditions enforced on a heuristic basis by Drugan and Rice (1984), Drugan and Shen (1987, 1990) and Leighton *et al.* (1987). Also, for the case of steady-state *weak* shocks, they proved the surprising result that the coefficients of the

Hugoniot function's Taylor expansion about the leading (or trailing) state of a steady smooth wave *vanish to all orders*, as opposed to vanishing only through second-order for general non-steady weak shocks. Although not *proving* that steady-state weak shocks are thermodynamically identical to steady smooth waves, since non-zero functions exist whose Taylor expansion coefficients all vanish, this result was nevertheless very suggestive.

In the present work we follow up on this suggestion, and prove in Section 3 that a steady-state but otherwise general shock wave in an arbitrary medium is thermodynamically identical to a steady-state smooth wave with the same start and end states as the shock. The restriction to steady-state shocks permits removal of the assumption that the shock is weak, since the analysis does not involve use of a Taylor expansion.

This proof that a steady-state shock is thermodynamically identical to a steadily propagating smooth wave means that restrictions imposed by constitutive requirements on steady-state shocks can be deduced by analyzing steadily propagating smooth waves: the same purely mechanical constitutive equations assumed to govern smooth waves must also, due to this thermodynamical equivalence, govern the material behavior within the shock. Further restrictions are imposed by the steady-state forms of the conservation laws and geometrical compatibility within the smooth wave. In particular, by combining these with constitutive restrictions weaker than the maximum plastic work inequality, we rigorously confirm in Section 4 the previous shock restrictions deduced by Drugan and Rice (1984), Drugan and Shen (1987, 1990) and Leighton *et al.* (1987), while showing that these restrictions apply to a broader class of materials than treated by these authors.

A special case of the group of situations in which steady-state elastic–plastic shock waves can be ruled out, except those propagating at elastic wave speeds, is that of plane strain elastic–ideally plastic materials satisfying plastic normality and whose yield surfaces do not contain linear portions. The specific derivation of this result is reviewed and extended in Section 5 via the present new approach. This conclusion has great practical importance in the analysis of dynamic steady-state elastic–plastic plane-strain crack growth: when such cracks propagate more slowly than the material's elastic wave speeds (as experiments appear to show they do), we conclude that there can be no shock waves attending propagation of a crack tip under such conditions. Yet the recent numerical finite element analysis of Varias and Shih (1994) seems to show "shocks" propagating with such a crack tip. However, their numerical procedure does not enforce the requirements we prove here must be satisfied by the path through a physically-acceptable shock. Our contention is that enforcement of these thermodynamics-mandated shock requirements is analogous to the fact that any physically correct elastic–plastic solution must involve only non-negative plastic work rate, as also required by the underlying laws of thermodynamics. As we will argue, the conclusion is that either there is a shock-free solution with features too subtle for Varias and Shih's (1994) numerical method to handle, or that no solution exists to the *steady-state* governing equations for this case, implying that steady-state plane-strain dynamic crack growth is not possible in this model (and potentially not in nature for materials with purely nonlinear yield surfaces, plastic normality and no hardening).

2. CONSERVATION LAWS AND THEIR STEADY-STATE FORMS FOR ARBITRARY MEDIA

In terms of a fixed reference configuration, the weak (integral) forms of conservation of mass, linear momentum, angular momentum and energy (First Law of thermodynamics), together with the Second Law of thermodynamics are, respectively:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} \rho_0 \,\mathrm{d}V_0 = 0 \tag{1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} \rho_0 \mathbf{v} \,\mathrm{d}V_0 = \int_{S_0} \mathbf{t}_0 \,\mathrm{d}S_0 + \int_{V_0} \rho_0 \mathbf{b} \,\mathrm{d}V_0 \tag{1b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} \mathbf{x}\times\rho_0 \mathbf{v} \,\mathrm{d}V_0 = \int_{S_0} \mathbf{x}\times\mathbf{t}_0 \,\mathrm{d}S_0 + \int_{V_0} \mathbf{x}\times\rho_0 \mathbf{b} \,\mathrm{d}V_0 \tag{1c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} \left(\frac{1}{2}\rho_0 \mathbf{v}\cdot\mathbf{v}+\rho_0 u\right)\mathrm{d}V_0 = \int_{S_0} \mathbf{t}_0\cdot\mathbf{v} \,\mathrm{d}S_0 + \int_{V_0} \rho_0 \mathbf{b}\cdot\mathbf{v} \,\mathrm{d}V_0 + \int_{V_0} \rho_0 r \,\mathrm{d}V_0 - \int_{S_0} \mathbf{Q}\cdot\mathbf{n}_0 \,\mathrm{d}S_0 \tag{1d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_0} \rho_0 \eta \,\mathrm{d}V_0 \geqslant \int_{V_0} \frac{\rho_0 r}{\theta} \,\mathrm{d}V_0 - \int_{S_0} \frac{\mathbf{Q}\cdot\mathbf{n}_0}{\theta} \,\mathrm{d}S_0. \tag{1e}$$

Here, $\mathrm{d}/\mathrm{d}t$ denotes material time rate; ρ_0 is reference mass density; V_0 is an arbitrary reference configuration subvolume having surface S_0 and unit outward normal $\mathbf{n}_0$; $\mathbf{v}$ is the material velocity vector; $\mathbf{t}_0$ is the First Piola–Kirchhoff (nominal) traction vector; $\mathbf{b}$ is the body force vector per unit reference mass; $\mathbf{x}$ is the current configuration position vector; $\times$ denotes vector cross product; $\cdot$ denotes vector inner product; u is internal energy, r is heat source and η is entropy, all per unit reference mass; $\mathbf{Q}$ is the nominal heat flux vector, and θ is the absolute temperature.

2.1. *Jump forms of the conservation laws*

When a shock wave is sufficiently narrow physically to be idealized as a jump discontinuity surface, the conservation laws assume well-known jump forms. Denote the reference configuration image of this discontinuity surface by Σ_0, with speed c_0 in the propagation direction of unit normal vector $\mathbf{N}$. The jump of any field variable ψ is denoted by double brackets as $[\![\psi]\!] \equiv \psi^+ - \psi^-$, where ψ^+ and ψ^- are the limiting values of ψ just ahead of and just behind the shock, respectively. An overbar denotes the average of the $(+)$ and $(-)$ side limiting values: $\bar{\psi} \equiv (\psi^+ + \psi^-)/2$. Application of the weak forms of the conservation laws (1) to a "pillbox"-shaped element containing this surface, in the limit as the element collapses onto the surface results in the jump conditions (see, e.g., Chadwick, 1976), respectively (conservation of angular momentum does not provide an additional restriction to (3)):

$$[\![\rho_0 c_0]\!] = 0 \tag{2}$$

$$[\![\mathbf{v}]\!] = -\frac{\mathbf{N}\cdot[\![\boldsymbol{\sigma}_0]\!]}{\rho_0 c_0} \tag{3}$$

$$[\![u]\!] = \frac{1}{\rho_0}\bar{\boldsymbol{\sigma}}_0^T : [\![\mathbf{F}]\!] + \frac{\mathbf{N}\cdot[\![\mathbf{Q}]\!]}{\rho_0 c_0} \tag{4}$$

$$[\![s]\!] - \frac{1}{\rho_0 c_0}\left[\!\!\left[\frac{\mathbf{Q}}{\theta}\right]\!\!\right]\cdot\mathbf{N} \leqslant 0. \tag{5}$$

Here, $\boldsymbol{\sigma}_0$ is the First Piola–Kirchhoff (nominal) stress tensor ($\mathbf{t}_0 = \mathbf{n}_0\cdot\boldsymbol{\sigma}_0$); $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient tensor, where $\mathbf{x}(\mathbf{X}, t)$ is the motion; a superscript T denotes tensor transpose; and : denotes tensor inner product, so that in terms of index notation with the summation convention, $\mathbf{A}:\mathbf{B} = A_{ij}B_{ij}$.

2.2. *Local (smooth) forms of the conservation laws*

In reference configuration regions where all field variables have continuous first derivatives with respect to both position and time, it is well-known that the integral forms (1) of the conservation laws reduce to the following local forms, respectively:

$$\dot{\rho}_0 = 0 \tag{6}$$

$$\nabla_0\cdot\boldsymbol{\sigma}_0 = \rho_0(\dot{\mathbf{v}} - \mathbf{b}) \tag{7}$$

$$\mathbf{F}\cdot\boldsymbol{\sigma}_0 = (\mathbf{F}\cdot\boldsymbol{\sigma}_0)^T \tag{8}$$

$$\rho_0\dot{u} = \boldsymbol{\sigma}_0^T : \dot{\mathbf{F}} + \rho_0 r - \nabla_0\cdot\mathbf{Q} \tag{9}$$

$$\dot{s} - \frac{r}{\theta} + \frac{1}{\rho_0}\nabla_0\cdot\left(\frac{\mathbf{Q}}{\theta}\right) \geqslant 0. \tag{10}$$

Here and in the following, a superposed dot denotes material time rate, and $\nabla_0\cdot$ denotes divergence with respect to reference configuration coordinates. To these local forms (6)–(10) we append the material time rate of the deformation gradient tensor definition, which can be regarded as the compatibility condition:

$$\dot{\mathbf{F}} = \dot{\mathbf{x}}\overleftarrow{\nabla}_0 = \mathbf{v}\overleftarrow{\nabla}_0. \tag{11}$$

Incidentally, although we have recorded the Second Law of thermodynamics here for completeness, we shall not explicitly use it in our analysis of general media, and it will be supplanted by the subsuming but stronger maximum plastic work inequality in our analysis of elastic–plastic materials.

2.3. *Steady-state local forms of the conservation laws*

The local forms of the conservation laws simplify when they are applied to the fields associated with a smooth wave that propagates under steady-state conditions with respect to an observer moving with the wave. Specifically, following Brannon *et al.* (1995), we will analyze a smooth wave whose reference configuration image propagates without rotating under steady-state conditions with constant speed c_0 in the direction of its unit normal $\mathbf{N}$. Under these conditions, the material time rate of any field quantity ψ within the wave is simply

$$\dot{\psi} = -c_0 \psi \tilde{\nabla}_0 \cdot \mathbf{N}, \tag{12}$$

and we have also required

$$\dot{\mathbf{N}} = \mathbf{0}, \quad \dot{c}_0 = 0. \tag{13}$$

Since our goal is to explore whether a smooth wave can emulate a shock, we further restrict consideration to smooth waves having the feature that field quantities may vary *through* the wave (i.e., in the $\mathbf{N}$-direction), but *not* parallel to the wave. The reference configuration del operator then simplifies to

$$\nabla_0 = \mathbf{N}\frac{\mathrm{d}}{\mathrm{d}\nu}, \tag{14}$$

where ν measures distance in the $\mathbf{N}$ direction.

Assuming that no body forces act, it is straightforward to show (see Brannon *et al.*, 1995) that (12) and (14) together give:

$$\nabla_0 \cdot \boldsymbol{\sigma}_0 = -\frac{1}{c_0}\dot{\boldsymbol{\sigma}}_0^T \cdot \mathbf{N}; \tag{15}$$

that (12) and (14) reduce compatibility (11) to the useful forms:

$$\dot{\mathbf{F}} \cdot \mathbf{N} = -\frac{1}{c_0}\dot{\mathbf{v}}, \tag{16}$$

$$\dot{\mathbf{F}} = (\dot{\mathbf{F}} \cdot \mathbf{N})\mathbf{N}; \tag{17}$$

and that (15) and (16) reduce linear momentum conservation (7) to

$$\mathbf{N} \cdot \dot{\boldsymbol{\sigma}}_0 = \rho_0 c_0^2 \dot{\mathbf{F}} \cdot \mathbf{N}. \tag{18}$$

It will also prove useful to note that taking the material time rate of (17) and (18), applying (6) and (13), gives respectively

$$\ddot{\mathbf{F}} = (\ddot{\mathbf{F}} \cdot \mathbf{N})\mathbf{N}, \tag{19}$$

$$\mathbf{N} \cdot \ddot{\boldsymbol{\sigma}}_0 = \rho_0 c_0^2 \ddot{\mathbf{F}} \cdot \mathbf{N}. \tag{20}$$

3. COMPARISON OF A SHOCK WITH A SMOOTH WAVE IN ARBITRARY MEDIA

3.1. *Assessment of weak shocks vs smooth waves via Taylor expansion of the Hugoniot function*

Here we summarize Brannon *et al.*'s (1995) generalization of the Courant and Friedrichs (1948) approach to comparison of a shock with a smooth wave. While Courant and Friedrich's analysis addressed only weak one-dimensional shocks in inviscid fluids, Brannon *et al.* showed that the basic idea of their analysis could be extended to general three-dimensional deformations of *arbitrary media*. For these general conditions, the Hugoniot function H is defined as

$$H(\mathbf{F},\boldsymbol{\sigma}_0,u,\mathbf{Q}) \equiv (u-u_1)-\frac{1}{2\rho_0}(\boldsymbol{\sigma}_0^T+\boldsymbol{\sigma}_{01}^T):(\mathbf{F}-\mathbf{F}_1)-\frac{\mathbf{N}\cdot(\mathbf{Q}-\mathbf{Q}_1)}{\rho_0 c_0}. \tag{21}$$

Regarding $\{\mathbf{F},\boldsymbol{\sigma}_0,u,\mathbf{Q}\}$ as the set of all states on one side of a shock front associated with a given STATE 1 $\{\mathbf{F}_1,\boldsymbol{\sigma}_{01},u_1,\mathbf{Q}_1\}$ on the other side, the jump (shock) form of the conservation of energy (4) is then expressed by the condition

$$H = 0. \tag{22}$$

To explore how a smooth wave differs from a shock, Courant and Friedrichs' idea was that one could evaluate the Hugoniot function for a smooth wave by a Taylor expansion, and then determine the degree to which it differs from the shock value (22). Brannon *et al.* (1995) generalized this by parameterizing the variables in the Hugoniot function as functions of time, and then performing a Taylor expansion of H about STATE 1 in a smooth wave:

$$H(t) = H_1+\dot{H}_1(t-t_1)+\frac{1}{2!}\ddot{H}_1(t-t_1)^2+\frac{1}{3!}\dddot{H}_1(t-t_1)^3+\ldots, \tag{23}$$

where the coefficients H_1, $\dot{H}_1$, etc. denote the values of H and its material time derivatives at STATE 1. Thus (23) applies at a fixed material point, with t_1 corresponding to the time when one side (STATE 1) of the wave has just reached that material point, and t is the time when the other side of the wave reaches the point. Such an expansion is sensible when one is exploring whether a smooth wave can emulate a *weak* shock, in which case one anticipates that $|t-t_1| \ll 1$ (when suitably nondimensionalized).

Assuming that the heat source is zero and the heat flux is constant (Courant and Friedrichs assumed they are both zero), Brannon *et al.* applied the smooth form of the First Law, (9), to show that the coefficients in the Taylor expansion have the following values for a *general non-steady* smooth wave:

$$H_1=\dot{H}_1=\ddot{H}_1\equiv 0,\quad \dddot{H}_1=\frac{1}{2\rho_0}(\boldsymbol{\sigma}_{01}^T:\ddot{\mathbf{F}}_1-\ddot{\boldsymbol{\sigma}}_{01}^T:\dot{\mathbf{F}}_1). \tag{24}$$

Thus, they showed that a weak shock is thermodynamically equivalent to a smooth wave until third order, but these do differ at third order since $\dddot{H}_1 \neq 0$ in general.

Brannon *et al.* (1995) also explored this comparison in the special case of steady-state shock and smooth wave propagation. In this case, the simplified restrictions due to momentum conservation and compatibility, (16)–(18), apply. As shown by Brannon *et al.*, the forms (19), (20) of these restrictions immediately require that the last of (24) becomes $\dddot{H}_1 \equiv 0$, and furthermore that all higher-order coefficients in the Taylor expansion (23) vanish also in this steady-state case. This suggests at least that the smooth wave approximation improves substantially for steady-state weak shocks, and carries the tantalizing implication that a smooth wave may be thermodynamically identical to a shock in this steady-state case. [That this is merely an implication and not yet a proof is due to the fact that smooth functions exist which are zero at a point and whose Taylor expansion coefficients all vanish at that point, but which nevertheless become nonzero away from that point; a simple example is $e^{-1/(t-t_1)^2}$, where e is the natural logarithm base.]

3.2. *Direct demonstration that a suitable smooth wave exactly models a steadily propagating shock*

Here we provide a new, direct proof that the conclusion hinted at by Brannon *et al.*'s (1995) analysis is indeed true: a shock propagating under steady-state conditions is thermodynamically identical to a suitably-constructed smooth wave. Furthermore, since the proof is direct in the sense of not involving use of the Taylor expansion (23), it is valid for shocks of *arbitrary strength*.

As reviewed in Section 3.1, Brannon *et al.* (1995) demonstrated that for general three-dimensional deformations of arbitrary materials, there is a thermodynamic difference between a shock and a smooth wave: this involves additional entropy production in the shock, and it shows up as a modification to the First Law of thermodynamics, so that the shock form of this law differs from the smooth wave form. Brannon *et al.* compared this difference by expressing the First Law for a shock as the requirement that the Hugoniot function vanish, and then quantified the difference between this and the smooth wave form by showing that if the Hugoniot function is expanded in a Taylor series for a smooth wave, its third-order coefficient is non-vanishing in general. However, their demonstration that all the Taylor series coefficients vanish in the case of steady-state propagation of smooth waves and shocks strongly suggests that under such conditions, the thermodynamic differences between shocks and smooth waves vanish, which would mean that the smooth wave form and the shock form of the First Law should reduce to the same condition in such steady-state situations.

We will show this now by directly integrating the smooth wave form of the First Law across a steady-state smooth wave, and verifying that the resulting condition is identical to the shock (jump) form of the First Law. From (9), the smooth form of the First Law is, assuming as before zero heat source and constant heat flux

$$\rho_0 \dot{u} = \boldsymbol{\sigma}_0^T : \dot{\mathbf{F}}. \tag{25}$$

We express integration of this at a fixed material point just from the time of arrival (t^+) to the time of departure (t^-) of a smooth wave as

$$\int_{t^+}^{t^-} \dot{u}\,\mathrm{d}t = \frac{1}{\rho_0}\int_{t^+}^{t^-} \boldsymbol{\sigma}_0^T : \dot{\mathbf{F}}\,\mathrm{d}t. \tag{26}$$

Note that for a general nonsteady wave, (26) is not integrable in general; i.e., more specific information is needed, and the result will differ for different waves. However, under the special conditions detailed in Section 2.3, namely steady-state propagation of a smooth wave with a non-rotating reference configuration image and field quantities that vary through the wave but not along it (and no body forces acting), (26) can be integrated directly for finite deformations and arbitrary materials: applying the compatibility requirement (17) under such steady-state conditions shows that

$$\boldsymbol{\sigma}_0^T : \dot{\mathbf{F}} = (\mathbf{N} \cdot \boldsymbol{\sigma}_0) \cdot (\dot{\mathbf{F}} \cdot \mathbf{N}). \tag{27}$$

Also recall we showed linear momentum conservation under these conditions to reduce to (18), the substitution of which for $\dot{\mathbf{F}} \cdot \mathbf{N}$ into (27) gives

$$\boldsymbol{\sigma}_0^T : \dot{\mathbf{F}} = \frac{1}{\rho_0 c_0^2} (\mathbf{N} \cdot \boldsymbol{\sigma}_0) \cdot (\mathbf{N} \cdot \dot{\boldsymbol{\sigma}}_0). \tag{28}$$

Recalling from (13) that $\dot{\mathbf{N}} = \mathbf{0}$, we temporarily define for convenience the vector $\mathbf{T} \equiv \mathbf{N} \cdot \boldsymbol{\sigma}_0$. Then using this and (28), (26) becomes

$$\int_{t^-}^{t^+} \dot{u}\, \mathrm{d}t = \frac{1}{\rho_0^2 c_0^2} \int_{t^-}^{t^+} \mathbf{T} \cdot \dot{\mathbf{T}}\, \mathrm{d}t. \tag{29}$$

This can now be integrated directly, giving

$$u^- - u^+ = \frac{1}{2\rho_0^2 c_0^2} [\mathbf{T}^- \cdot \mathbf{T}^- - \mathbf{T}^+ \cdot \mathbf{T}^+] = -\frac{1}{2\rho_0^2 c_0^2} (\mathbf{T}^+ + \mathbf{T}^-) \cdot (\mathbf{T}^+ - \mathbf{T}^-). \tag{30}$$

Next, integrating linear momentum conservation (18) and compatibility (17) at a fixed material point just during passage of the smooth wave, applying (6) and (13) gives, respectively:

$$(\mathbf{T}^+ - \mathbf{T}^-) = \rho_0 c_0^2 (\mathbf{F}^+ - \mathbf{F}^-) \cdot \mathbf{N}, \tag{31}$$

$$(\mathbf{F}^+ - \mathbf{F}^-) = [(\mathbf{F}^+ - \mathbf{F}^-) \cdot \mathbf{N}]\mathbf{N}. \tag{32}$$

Application of first (31) and then (32) to (30) leads to the final result:

$$u^+ - u^- = \frac{1}{2\rho_0} \mathbf{N} \cdot (\boldsymbol{\sigma}_0^- + \boldsymbol{\sigma}_0^+) \cdot (\mathbf{F}^+ - \mathbf{F}^-) \cdot \mathbf{N} = \frac{1}{2\rho_0} (\boldsymbol{\sigma}_0^+ + \boldsymbol{\sigma}_0^-)^T : (\mathbf{F}^+ - \mathbf{F}^-). \tag{33}$$

Observe that this is identical to the shock (jump) form (4) when, as assumed here, heat flux is constant:

$$[\![u]\!] = \frac{1}{\rho_0} \bar{\boldsymbol{\sigma}}_0^T : [\![\mathbf{F}]\!]. \tag{34}$$

Thus we have shown directly that when the smooth form of the First Law of thermodynamics is integrated at a fixed material point just during passage of a steady-state non-rotating smooth wave of the type that emulates a shock wave (i.e., having field quantities that vary through the wave but not along it, since for a shock any change in field quantities along the shock is negligible compared to their change across it), the result is *exactly* the *shock* form of the First Law. That is, we have proven that the thermodynamic difference that Brannon *et al.* (1995) showed exists between a general non-steady shock and a smooth wave *vanishes completely* for steady-state shock propagation, when the smooth wave is chosen as described.

We thus conclude the following from these new results and those of Brannon *et al.*

(1995): for general nonsteady shocks, when such a shock is weak (i.e., involves only small jumps in field variables), it can be very accurately *approximated* by a suitably-chosen smooth wave, since Brannon *et al.* proved that these differ thermodynamically only at third order in the Hugoniot function Taylor expansion (23). However, for steady-state shock propagation, we have just proved that a suitably-chosen smooth wave is a thermodynamically *exact* representation of the shock, for shocks of *arbitrary strength* (i.e., not limited to weak shocks). Our contention is thus that *restrictions on the existence and allowable types of steady-state shocks can be derived exactly by analyzing a suitably-chosen smooth wave*, by using the steady-state forms of the governing equations reviewed in Section 2.3 accompanied by, for specific material types, purely mechanical constitutive restrictions. Furthermore, such an approach provides *approximate* restrictions on weak, slightly nonsteady shocks. These conclusions validate the previous analyses of shock restrictions reviewed in Section 1, and lead to the new, stronger restrictions derived below.

4. STRENGTHENED RESTRICTIONS ON STEADY ELASTIC–PLASTIC SHOCKS

The fact proved in Section 3.2, that a shock wave which propagates under steady-state, non-rotating conditions is thermodynamically identical to a suitably-chosen smooth wave, rigorously legitimizes the derivation of restrictions on such a shock by analysis of an appropriate smooth wave within a purely mechanical constitutive model. This will now be performed for steady-state shocks in elastic–plastic media satisfying only skeletal constitutive restrictions. Drugan and Rice (1984) and Drugan and Shen (1987, 1990) have previously carried out such analyses for quasi-static and dynamic shocks, by making assumptions tantamount to those proved rigorously in Section 3.2 here; we show now that the explicit recognition of this steady shock–smooth wave equivalence leads to a strengthening of the restrictions derived in these references, probably to the strongest forms possible.

For simplicity and clarity, we will perform the analyses in this section within a small-displacement-gradient framework, so that the results obtained will explicitly generalize those derived in Drugan and Rice (1984) and Drugan and Shen (1987); the finite deformation versions of these results are obtained by a fairly straightforward application of the approach outlined here to the formulation laid out in Drugan and Shen (1990), using the results of the present Section 3.2.

4.1. *Steady-state smooth forms of conservation laws within a small-displacement-gradient formulation*

For the remainder of the paper, we will employ a small-displacement-gradient framework. Within this, the smooth forms of the needed conservation laws and compatibility are, when no body forces act:

$$\dot{\rho} = 0 \tag{35}$$

$$\nabla \cdot \boldsymbol{\sigma} = \rho \dot{\mathbf{v}} \tag{36}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{37}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\mathbf{u}\bar{\nabla} + (\mathbf{u}\bar{\nabla})^T], \tag{38}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\varepsilon}$ the infinitesimal strain tensor, $\mathbf{u}$ the displacement vector and ρ is mass density.

The conditions (12)–(14) describing the steady-state, constant-speed propagation of a smooth, non-rotating wave with the shock-emulating property that quantities vary through but not along it become:

$$\dot{\psi} = -c\psi\bar{\nabla}\cdot\mathbf{n} \tag{39}$$

$$\dot{\mathbf{n}} = \mathbf{0}, \quad \dot{c} = 0 \tag{40}$$

$$\nabla = \mathbf{n}\frac{\mathrm{d}}{\mathrm{d}\nu}, \tag{41}$$

where the wave propagates with speed c in the direction of its unit normal vector $\mathbf{n}$ and ν measures distance in the $\mathbf{n}$ direction. Combining (39) and (41) shows

$$\dot{\psi} = -c\frac{\mathrm{d}\psi}{\mathrm{d}\nu}, \tag{42}$$

and application of this and (41) produces

$$\nabla\cdot\boldsymbol{\sigma} = \mathbf{n}\cdot\frac{\mathrm{d}\boldsymbol{\sigma}}{\mathrm{d}\nu} = -\frac{1}{c}\mathbf{n}\cdot\dot{\boldsymbol{\sigma}}. \tag{43}$$

Thus, linear momentum conservation (36) reduces via (43) to

$$\mathbf{n}\cdot\dot{\boldsymbol{\sigma}} = -\rho c\dot{\mathbf{v}}, \tag{44}$$

which in incremental form at a fixed material point is

$$\boxed{\mathbf{n}\cdot\mathrm{d}\boldsymbol{\sigma} = -\rho c\,\mathrm{d}\mathbf{v}.} \tag{45}$$

Next, choosing $\psi = \mathbf{u}$ in (42) produces

$$\mathbf{v} = \dot{\mathbf{u}} = -c\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\nu}; \tag{46}$$

taking the increment of this at a fixed material point gives, using (40)

$$\mathrm{d}\mathbf{v} = -c\,\mathrm{d}\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\nu}\right). \tag{47}$$

Using (41), (38) is

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\nu}\mathbf{n} + \mathbf{n}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\nu}\right], \tag{48}$$

the increment of which at a fixed material point becomes, using (40) and (47):

$$\boxed{\mathrm{d}\boldsymbol{\varepsilon} = -\frac{1}{2c}[(\mathrm{d}\mathbf{v})\mathbf{n}+\mathbf{n}(\mathrm{d}\mathbf{v})].} \tag{49}$$

It is interesting to observe that (45) and (49) are identical to the forms *assumed*, on the basis of heuristic arguments, to apply to a material particle during shock passage by Drugan and Shen (1987) and Leighton *et al.* (1987), and in the quasi-static case with the right-hand side of (45) equal to zero by Drugan and Rice (1984). We have shown here that these arise rigorously for a material particle during steady-state smooth wave passage, and that a suitably chosen smooth wave rigorously represents a steady-state shock.

4.2. *Skeletal elastic–plastic constitutive assumptions*

Here we summarize the skeletal set of constitutive assumptions employed by Drugan and Rice (1984) and Drugan and Shen (1987) to represent a very broad class of rate-independent elastic–plastic materials of practical significance for which nevertheless strong shock restrictions can be deduced. When these are employed in the following section to derive shock restrictions by analyzing a smooth wave, we shall be able to relax some of the constitutive restrictions required by these authors, and thus obtain results of far greater (probably the greatest) generality.

Total strain increments are assumed to decompose additively into elastic and plastic parts, with the elastic part being linearly related to an increment in the stress tensor by the fourth-rank, positive–definite elastic compliance tensor $\mathbf{M}$, whose components possess the usual symmetries $M_{ijkl} = M_{jikl} = M_{klij}$:

$$\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^e + \mathrm{d}\boldsymbol{\varepsilon}^p = \mathbf{M} : \mathrm{d}\boldsymbol{\sigma} + \mathrm{d}\boldsymbol{\varepsilon}^p. \tag{50}$$

Plastic deformation is constrained to satisfy the maximum plastic work inequality:

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^0) : \mathrm{d}\boldsymbol{\varepsilon}^p \geqslant 0, \tag{51}$$

where $\boldsymbol{\sigma}$ and $\mathrm{d}\boldsymbol{\varepsilon}^p$ are the actual current stress and plastic strain increment, and $\boldsymbol{\sigma}^0$ is any other stress state lying on or within the current yield surface.

4.3. *Restrictions on steadily propagating, non-rotating shock waves*

We now employ the result proven in Section 3.2 that a shock propagating under steady-state, non-rotating conditions is thermodynamically identical to a steadily propagating, non-rotating smooth wave having the same start and end states as the shock and with the feature that field quantities may in general vary through the wave but not along it. Thus we may derive restrictions on such steady shock waves by analyzing a smooth wave of the type just described, their thermodynamical equivalence rigorously justifying use of a purely mechanical constitutive model in the smooth wave analysis. This means that to determine whether a field quantity can jump across a shock, we must analyze whether it can accumulate a change across a smooth wave. The minimum necessary condition for such an accumulation is that if one monitors a fixed material point during passage of a smooth wave, the field quantity must experience an incremental change at some time increment during wave passage. The weakest way to enforce the maximum plastic work inequality (i.e., the way leading to the greatest generality of results) is thus to choose $\boldsymbol{\sigma}^0$ to be the stress state at a fixed material point during smooth wave passage at the instant before the current stress $\boldsymbol{\sigma}$ is attained; with this choice, (51) becomes

$$\mathrm{d}\boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}^p \geqslant 0, \tag{52}$$

which restricts the plastic response only by requiring it to be non-softening and consistent with normality of plastic strain increments to the yield surface; beyond this, any type of material hardening, including nonhardening (ideally plastic) behavior, is permitted.

The analysis of the restrictions imposed by (52) will be facilitated by eliminating $\mathrm{d}\boldsymbol{\varepsilon}^p$ from it via (50):

$$\mathrm{d}\boldsymbol{\sigma} : (\mathrm{d}\boldsymbol{\varepsilon} - \mathbf{M} : \mathrm{d}\boldsymbol{\sigma}) \geqslant 0. \tag{53}$$

The implications of this become more clear when the steady-state compatibility condition (49) is used to substitute for $\mathrm{d}\boldsymbol{\varepsilon}$, so that (53) becomes

$$-\frac{1}{2c}\,\mathrm{d}\boldsymbol{\sigma} : [(\mathrm{d}\mathbf{v})\mathbf{n} + \mathbf{n}(\mathrm{d}\mathbf{v})] - \mathrm{d}\boldsymbol{\sigma} : \mathbf{M} : \mathrm{d}\boldsymbol{\sigma} \geqslant 0. \tag{54}$$

By making use of angular momentum conservation (37), this reduces to

$$-\frac{1}{c}\,\mathbf{n} \cdot \mathrm{d}\boldsymbol{\sigma} \cdot \mathrm{d}\mathbf{v} - \mathrm{d}\boldsymbol{\sigma} : \mathbf{M} : \mathrm{d}\boldsymbol{\sigma} \geqslant 0, \tag{55}$$

where the equality applies for nonhardening material.

4.3.1. *Restrictions on steady-state shock waves in quasi-static deformations.* Here we analyze which field quantities can accumulate changes during steady-state smooth wave passage in quasi-static deformations, and thereby determine restrictions on steady-state quasi-static shock waves. As noted earlier, for quasi-static deformations, material inertia is negligible, and thus the steady-state form of linear momentum conservation (45) reduces to

$$\mathbf{n} \cdot \mathrm{d}\boldsymbol{\sigma} = \mathbf{0}. \tag{56}$$

Substitution of this into (55) results in the requirement

$$-\mathrm{d}\boldsymbol{\sigma} : \mathbf{M} : \mathrm{d}\boldsymbol{\sigma} \geqslant \mathbf{0}. \tag{57}$$

Since $\mathbf{M}$ is positive-definite, this immediately shows that

$$\mathrm{d}\boldsymbol{\sigma} = 0; \tag{58}$$

that is, no stress tensor changes can accumulate during the passage of a steady-state smooth wave. Application of this result to (50) shows that no elastic strain changes can accumulate either. Finally, steady-state compatibility (49) shows that no components of strain lying in the plane parallel to the smooth wave at any point can accumulate changes; that is, the only possible components of strain that can change across the wave are $\mathbf{n} \cdot \boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}^p \cdot \mathbf{n}$. Thus, we conclude that $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}^e$ cannot jump across a steady-state shock wave in quasi-static deformations, and at most only the $\mathbf{n}\cdot\boldsymbol{\varepsilon}^p$ components of strain can. These conclusions are identical to those derived by Drugan and Rice (1984), but we have here shown them to apply to all elastic–plastic materials that satisfy (50) and (52), which is a much broader constitutive class than that to which Drugan and Rice's analysis applies, as discussed further in Section 4.4.

4.3.2. *Restrictions on steadily propagating shock waves in dynamic deformations.* For dynamic deformations, the full form (45) of linear momentum conservation applies; using this to substitute for $\mathrm{d}\mathbf{v}$, (55) becomes:

$$\boxed{(\mathbf{n}\cdot\mathrm{d}\boldsymbol{\sigma})\cdot(\mathbf{n}\cdot d\boldsymbol{\sigma})-\rho c^2\,\mathrm{d}\boldsymbol{\sigma}:\boldsymbol{\mu}\,\mathrm{d}\boldsymbol{\sigma}\geqslant 0.} \tag{59}$$

This inequality restricts, but does not rule out, stress changes across a smooth wave.

In addition to (59), in order that $\mathrm{d}\boldsymbol{\sigma}\neq\mathbf{0}$ during any part of a smooth wave passage for wave speeds not equal to an elastic wave speed, the stress state must remain at yield. This is proved by contradiction: Suppose that $\mathrm{d}\boldsymbol{\sigma}$ takes (or maintains) the stress state below yield. Then, the corresponding strain increment must be purely elastic, so that from (50)

$$\mathrm{d}\boldsymbol{\varepsilon}=\mathrm{d}\boldsymbol{\varepsilon}^e=\mathbf{M}:\mathrm{d}\boldsymbol{\sigma}\quad\Rightarrow\quad\mathrm{d}\boldsymbol{\sigma}=\mathbf{C}:\mathrm{d}\boldsymbol{\varepsilon}, \tag{60}$$

where $\mathbf{C}=\mathbf{M}^{-1}$ is the elastic modulus tensor. Taking the dot product of (60) with $\mathbf{n}$ gives

$$\mathbf{n}\cdot\mathrm{d}\boldsymbol{\sigma}=\mathbf{n}\cdot\mathbf{C}:\mathrm{d}\boldsymbol{\varepsilon}, \tag{61}$$

which may be rewritten using the steady-state forms of momentum conservation (45) and compatibility (49), together with the symmetries of $\mathbf{C}$ as

$$(\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n}-\rho c^2\mathbf{I})\cdot\mathrm{d}\mathbf{v}=\mathbf{0}, \tag{62}$$

where $\mathbf{I}$ is the identity tensor. This permits $\mathrm{d}\mathbf{v}\neq\mathbf{0}$ only when

$$\det(\mathbf{n}\cdot\mathbf{C}\cdot\mathbf{n}-\rho c^2\mathbf{I})=0, \tag{63}$$

the three solutions for c of this are the elastic wave speeds, and (49) and (60) show that $\mathrm{d}\mathbf{v}\neq\mathbf{0}$ is needed for $\mathrm{d}\boldsymbol{\sigma}\neq\mathbf{0}$. Thus, we conclude that $\mathrm{d}\boldsymbol{\sigma}$ must be such that $\boldsymbol{\sigma}$ remains at yield for $\mathrm{d}\boldsymbol{\sigma}\neq\mathbf{0}$ during passage of a smooth wave whose propagation speed differs from an elastic wave speed.

The results of this section, together with those of Section 3.2, show that in order for a steady-state shock wave whose propagation speed c is not an elastic wave speed [i.e., does not satisfy (63)] to exhibit a nonzero stress jump during a dynamic deformation, a steady-state smooth wave must exist that exhibits, at least during some portion of the wave, a nonzero stress increment which *both* (i) satisfies (59) *and* (ii) corresponds to $\boldsymbol{\sigma}$ remaining at yield. These results are in accord with those derived by Drugan and Shen (1987), but we have here shown them to apply to all elastic–plastic materials that satisfy (50) and (52), which is a much broader constitutive class than that analyzed by Drugan and Shen, as discussed further in Section 4.4.

For the case of nonhardening materials, stronger restrictions emerge. Nonhardening response, together with the just-proved requirement that the stress state must remain at yield, corresponds to an *equality* in (52), which carries through the above analysis so that (59) becomes

$$(\mathbf{n}\cdot d\boldsymbol{\sigma})\cdot(\mathbf{n}\cdot d\boldsymbol{\sigma}) - \rho c^2 \, d\boldsymbol{\sigma}:\mathbf{M}:d\boldsymbol{\sigma} = 0. \tag{64}$$

This restriction allows one immediately to rule out the possibility of steady-state shock waves in broad classes of dynamic deformations. For example, for anti-plane strain deformations having the only nonzero stress components be $\sigma_{13}(x_1, x_2) = \sigma_{31}$, $\sigma_{23}(x_1, x_2) = \sigma_{32}$, and choosing x_1 to lie in the **n**-direction, (64) reduces to

$$(d\sigma_{13})^2 - 4\rho c^2[M_{1313}(d\sigma_{13})^2 + 2M_{1323}\, d\sigma_{13}\, d\sigma_{23} + M_{2323}(d\sigma_{23})^2] = 0. \tag{65}$$

Regarding this as a quadratic equation for $d\sigma_{23}$, one finds

$$d\sigma_{23} = \frac{-1}{M_{2323}}\left\{M_{1323} \pm \left[M_{1323}^2 + M_{2323}\left(\frac{1}{4\rho c^2} - M_{1313}\right)\right]^{1/2}\right\} d\sigma_{13}. \tag{66}$$

This requires σ_{13} to vary *linearly* with σ_{23} across any potential steady-state shock wave, but as noted above these must also satisfy yield through the shock. Thus, unless the yield condition contains a linear segment with which (66) coincides, no stress jump can accumulate unless it propagates at an elastic wave speed. This means, via (45) and (49), that no strain nor velocity jumps can accumulate either. This corroborates the conclusion reached by Drugan and Shen (1987), valid for arbitrary anisotropy, and generalizes it to nonhardening materials satisfying only positive-definiteness of **M** and plastic normality (i.e. removing Drugan and Shen's restriction to materials with convex yield surfaces). One can similarly employ (64) within the present approach to rule out steady-state shock waves in dynamic incompressible plane strain deformations of arbitrarily anisotropic materials, unless the yield surface contains a linear segment or the shock propagates at an elastic wave speed, by following a similar procedure to Drugan and Shen's for this case, thus again confirming and generalizing their finding. In Section 5 we will employ the present approach explicitly to analyze steady-state shock waves in dynamic plane strain deformations of isotropic Prandtl–Reuss–Mises material.

4.4. *Discussion: comparison with previous shock analyses*

The results of Section 4.3 rigorously confirm, for the case of steady-state nonrotating shock wave propagation, the shock restrictions derived previously by Drugan and Rice (1984) and Drugan and Shen (1987), while also proving that they apply to a far broader constitutive class than previously shown. Here, we have shown that they apply to all elastic–plastic materials that satisfy (50), (52) and positive-definiteness of **M**, while these previous studies were further restricted to materials satisfying (51) and which, if they harden, do so such that the current yield locus always contains all prior yield loci. The reason for these additional constitutive restrictions was that these authors arrived by heuristic argument at the requirements that the stress and deformation paths experienced by a material particle during shock passage had to satisfy (45), (49) and (50), and they applied these to *integrate* the maximum plastic work inequality (51) at a fixed material point just during shock passage:

$$\int_{\varepsilon^{p-}}^{\varepsilon^{p}} (\boldsymbol{\sigma}-\boldsymbol{\sigma}^{0}):\mathrm{d}\boldsymbol{\varepsilon}^{p} \geqslant 0, \tag{67}$$

where except for nonhardening materials they made the choice $\boldsymbol{\sigma}^0 = \boldsymbol{\sigma}^+$, the (constant) stress state at the material point just before shock arrival. With this choice of $\boldsymbol{\sigma}^0$, the restrictions produced by (67) are only valid provided $\boldsymbol{\sigma}^+$ remains on or within the yield surface during shock passage, which can be guaranteed in general only by restricting the material class to those whose current yield locus always contains all prior yield loci. This is not the optimal choice for $\boldsymbol{\sigma}^0$; i.e., it does not lead to restrictions for the broadest possible material class.

By contrast, our proof here (Section 3.2) that a steady-state shock can be exactly represented by a steady-state smooth wave facilitates examining *incremental* accumulation of field variable changes during passage of any infinitesimal portion of the smooth wave, as a necessary condition for a jump in such a field variable across a shock. This permits us to choose $\boldsymbol{\sigma}^0$ to be different for each time increment at a fixed material point during smooth wave passage [which we have done by choosing it to be the stress state at the time increment $\mathrm{d}t$ before the (arbitrary) current state $\boldsymbol{\sigma}$], which is the best possible way to choose $\boldsymbol{\sigma}^0$ and hence leads to the present results which apply to the widest possible constitutive class, delineated by the assumptions (50), (52) and positive-definiteness of $\mathbf{M}$.

5. APPLICATION: AN OPEN QUESTION IN DYNAMIC FRACTURE MECHANICS

An important set of applications for the shock wave restrictions derived previously by Drugan and Rice (1984), Drugan and Shen (1987, 1990) and Leighton *et al.* (1987), which we have rigorously substantiated and generalized here, has been to determine whether shock surfaces can accompany a growing crack in elastic–plastic solids. For example, the demonstrations by Drugan and Rice (1984) and Drugan and Shen (1990) in the "small strain" and finite strain cases, respectively, that no moving surfaces of stress discontinuity or of normal velocity discontinuity can exist in quasi-static deformations of elastic–plastic materials satisfying the maximum plastic work inequality enabled Rice and Nikolic (1985), Rice (1987), Ponte Castaneda (1987), Drugan and Chen (1989) and Reid and Drugan (1993), among others, to enforce stress and normal velocity continuity in their analyses of near-tip stress and deformation fields for quasi-statically growing cracks for a variety of material types and, in the last reference cited, within a finite deformation formulation. Dynamic elastic–plastic crack growth solutions have been far more problematic, as will be discussed further below, but one very interesting solution is that of Nikolic and Rice (1988), who analyzed near-tip stress and deformation fields for dynamic elastic–ideally plastic anti-plane shear crack growth in ductile single crystals. These authors applied the dynamic shock restrictions derived by Drugan and Shen (1987), which demand that for subsonic crack growth, a shock may accompany the growing crack tip only if the stress transition across the shock lies on a linear portion of the yield surface (the yield surface for the single crystals modeled *does* have linear portions). Nikolic and Rice were able to show that a dynamically growing crack in such ductile single crystals *must* be accompanied by an (antisymmetrical) pair of shock surfaces emanating from the crack tip, whose orientations depend on the crack propagation speed.

For either anti-plane shear or incompressible plane strain dynamic subsonic crack growth, in nonhardening materials that satisfy the maximum plastic work inequality and whose yield surfaces do *not* contain linear portions, the analysis of Drugan and Shen (1987) [and, for the special case of incompressible plane strain deformations of isotropic Prandtl–Reuss–Mises material, the independent analysis of Leighton *et al.* (1987)] shows that *a shock wave cannot exist at a dynamically propagating crack tip.* In the analyses just cited, some of the conditions employed to derive these conclusions were based on heuristic arguments, as we have noted explicitly above. However, in the present work for the case of steady-state nonrotating shock propagation, we have replaced those heuristic arguments with rigorous analysis.

In this section, we first specialize the rigorous analysis of Section 4.3.2 to the case of dynamic plane strain deformations of *nonhardening*, incompressible elastic–plastic materials whose yield surfaces do not contain linear portions. This analysis confirms the conclusions of Drugan and Shen (1987) and Leighton *et al.* (1987) that no shocks with speeds different from an elastic wave speed can exist under such conditions. Leighton *et al.* (1987) employed this conclusion to derive a shock-free asymptotic field near a dynamically growing crack in incompressible plane strain elastic–ideally plastic material. That paper also provides an excellent review of previous asymptotic analytical studies of dynamic elastic–plastic crack growth. We next confirm the ideas of Shen and Drugan (1990) to show the same must be true for *compressible* plane strain dynamic deformations. Finally, we discuss the implications of this on a recently-published numerical finite element analysis of dynamic plane strain elastic–ideally plastic crack growth.

5.1. *Restrictions on steady-state shock waves in dynamic plane strain deformations of nonhardening Prandtl–Reuss–Mises material*

Here we specialize the analysis of Section 4.3.2 to plane strain deformations of nonhardening isotropic Prandtl–Reuss–Mises material, for simplicity of illustration and because the specific dynamic crack growth problem to be discussed concerns this material. Thus the yield condition and flow rule take the forms

$$f(\boldsymbol{\sigma}) = \frac{1}{2}\mathbf{s}:\mathbf{s}-k^2 = 0 \tag{68}$$

$$\mathrm{d}\boldsymbol{\varepsilon} = \frac{1+\nu}{E}\mathrm{d}\boldsymbol{\sigma} - \frac{\nu}{E}\mathbf{I}\operatorname{tr}(\mathrm{d}\boldsymbol{\sigma}) + \mathrm{d}\Lambda\mathbf{s}, \tag{69}$$

where $\mathbf{s}$ is the deviatoric stress tensor, k is the (constant) yield stress in pure shear, E is Young's modulus, ν is Poisson's ratio, and $\mathrm{d}\Lambda \geqslant 0$ is constitutively unspecified. We introduce a Cartesian coordinate system such that for the plane strain deformations under consideration, x_3 is perpendicular to the plane of deformation, and x_1 is parallel to the smooth wave normal $\mathbf{n}$ at the material point under consideration. Thus,

$\boldsymbol{\sigma} = \boldsymbol{\sigma}(x_1, x_2)$, and $\sigma_{13} = \sigma_{23} \equiv 0$. For this choice of coordinate system and isotropic materials, (64) reduces to

$$\mathrm{d}\sigma_{1i}\,\mathrm{d}\sigma_{1i} - \frac{\rho c^2}{E}\left[(1+\nu)\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\sigma_{ij} - \nu(\mathrm{d}\sigma_{kk})^2\right] = 0, \tag{70}$$

where here and henceforth Latin subscripts have range 1, 2, 3 and the Einstein summation convention is employed.

The condition of plastic incompressibility is built into (69), taking the trace of which thus gives

$$\mathrm{d}\varepsilon_{kk} = \frac{1-2\nu}{E}\,\mathrm{d}\sigma_{kk}. \tag{71}$$

Using (49) with (45) to substitute for $\mathrm{d}\varepsilon_{kk}$, (71) becomes:

$$\mathrm{d}\sigma_{11} = \rho c^2\,\frac{1-2\nu}{E}\,\mathrm{d}\sigma_{kk}. \tag{72}$$

The plane strain condition requires, using (69):

$$\mathrm{d}\varepsilon_{33} = \frac{1}{E}\left[\mathrm{d}\sigma_{33} - \nu(\mathrm{d}\sigma_{11} + \mathrm{d}\sigma_{22})\right] + \mathrm{d}\Lambda s_{33} = 0. \tag{73}$$

For fully incompressible material response ($\nu = 1/2$) or when plastic strain increments are very much larger than elastic ones, (73) reduces to the well-known condition $s_{33} = 0$, i.e.,

$$\sigma_{33} = \tfrac{1}{2}(\sigma_{11} + \sigma_{22}). \tag{74a}$$

When $\nu \neq 1/2$ and plastic strain increments do not overwhelm elastic ones, and recalling that we have shown above that a subsonic shock cannot exist if the strain increments are purely elastic, the only other possibility is that $s_{33} \neq 0$, in which case (73) can be solved for $\mathrm{d}\Lambda$:

$$\mathrm{d}\Lambda = -\frac{\mathrm{d}\sigma_{33} - \nu(\mathrm{d}\sigma_{11} + \mathrm{d}\sigma_{22})}{E s_{33}}. \tag{74b}$$

Case 1. Let us first examine the case of fully incompressible material ($\nu = 1/2$). Then (72) and (74a) give, using the first to simplify the second:

$$\mathrm{d}\sigma_{11} = 0, \quad \mathrm{d}\sigma_{33} = \frac{1}{2}\,\mathrm{d}\sigma_{22}. \tag{75}$$

Using these, (70) specializes to

$$(\mathrm{d}\sigma_{12})^2\left[1 - \frac{3\rho c^2}{E}\right] = \frac{3\rho c^2}{4E}(\mathrm{d}\sigma_{22})^2, \tag{76}$$

which again applying (75) may be rewritten as

$$d\sigma_{12} = \pm\frac{1}{2}\left[\frac{3\rho c^2}{E-3\rho c^2}\right]^{1/2} d(\sigma_{11}-\sigma_{22}). \tag{77}$$

This shows that σ_{12} must vary linearly with $(\sigma_{11}-\sigma_{22})$ across a smooth wave, but in addition yield (68) in this case requires these to be related as, using (74a):

$$\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2 = k^2. \tag{78}$$

Thus (77), (78) and (75) show that no stress changes can occur in this case across a steadily propagating smooth wave that is not propagating at an elastic wave speed. Applying this result to (45) shows that no components of material velocity can change across such a steadily propagating smooth wave, which leads to the conclusion from (49) that no components of strain can change either. Hence, by our earlier analysis, no stress, strain or velocity jumps can occur across a steadily propagating shock. This rigorously confirms the conclusion reached independently by Drugan and Shen (1987) and Leighton *et al.* (1987), who employed the heuristic arguments mentioned earlier.

Case 2. Next we analyze the elastically compressible case ($\nu \neq 1/2$) when plastic strain increments are very much larger than elastic ones. A specific analysis of this case has not previously been published, to my knowledge. Now, (72) replaces the first of (75), but (74a) still applies, so that (72) reduces to

$$d\sigma_{11} = \frac{\rho c^2}{2K-\rho c^2}\, d\sigma_{22}, \tag{79}$$

where $K = E/[3(1-2\nu)]$ is the bulk modulus. Employing (79) and (74a), (70) reduces to

$$d\sigma_{11} = \pm\left\{\frac{\rho c^2[E-2(1+\nu)\rho c^2]}{(K-\rho c^2)[(5-4\nu)K-2(1+\nu)\rho c^2]}\right\}^{1/2} d\sigma_{12}. \tag{80}$$

Finally, using first (79) and then (80) results in

$$d(\sigma_{11}-\sigma_{22}) = 2\frac{\rho c^2-K}{\rho c^2}\, d\sigma_{11} = \pm 2\left\{\frac{(K-\rho c^2)[E-2(1+\nu)\rho c^2]}{\rho c^2[(5-4\nu)K-2(1+\nu)\rho c^2]}\right\}^{1/2} d\sigma_{12}, \tag{81}$$

showing that σ_{12} must vary linearly with $(\sigma_{11}-\sigma_{22})$ across a smooth wave. However, yield in form (78) also must be satisfied through the wave, showing again that no stress changes can accumulate unless the wave propagates at an elastic wave speed. This leads for the same reasons as in Case 1 to the conclusions that therefore no strain nor velocity components can vary either, and thus that a steadily propagating shock wave is not possible at non-elastic wave speeds.

Case 3. The final possibility is the situation in which the material is elastically compressible and plastic strain increments are of the same order of magnitude as elastic ones through a smooth wave. In this case, the plane strain condition (74a) that applied in the previous two cases must be replaced by (74b). This leads to an independent restriction on stress variations by substituting it into the 12 component of the flow rule (69), and then employing (49) and (45). The other equations restricting possible stress variations across a steady-state smooth wave in this case are (70), (72) and yield (68). The analysis of this equation system is quite complicated and will not be carried out here. However, Shen and Drugan (1990) integrated a set of equations equivalent to these across what amounts to an arbitrary portion of a steadily propagating smooth wave. By considering the geometrical shapes of the resulting conditions in stress space, they concluded that these conditions could simultaneously be satisfied at best at a set of discrete points in stress space, so that no continuous path for stress change would be possible across the wave. Again, as above, conditions (45) and (49) then lead to the conclusions that no components of strain nor velocity could change across the wave, and thus by the foregoing analysis here, a steadily propagating shock is again ruled out in this case unless it propagates at an elastic wave speed.

5.2. *Implications for plane strain steady-state dynamic crack propagation in elastic–ideally plastic Prandtl–Reuss–Mises material*

One specific implication of the results proved above, which as noted provide rigorous confirmation of conclusions reached previously by Drugan and Shen (1987), Leighton *et al.* (1987) and Shen and Drugan (1990), is that the stress, strain and velocity fields attending *steady-state* dynamic plane strain growth of a crack in elastic–ideally plastic Prandtl–Reuss–Mises material *must be fully continuous* for all crack propagation speeds below the material's elastic wave speeds. That is, our analysis rules out the possibility of a shock moving with such a growing crack.

Surprisingly, then, a recent numerical finite element analysis by Varias and Shih (1994) [henceforth abbreviated VS] of *steady-state* dynamic plane strain crack growth in both hardening and nonhardening Prandtl–Reuss–Mises materials appears to show a "shock" propagating with the crack tip when the crack growth speed is sufficiently high (but still well below elastic wave speeds). These results are directly at variance with the conclusions obtained above. Let us examine the particular one of their solutions for which this disagreement is easiest to see: In Figure 2(c) of VS, their solution is displayed for the strain field near a dynamically propagating crack tip with propagation speed of 0.2 times the elastic shear wave speed in nonhardening, essentially incompressible ($\nu = 0.49$) Prandtl–Reuss–Mises material. This solution exhibits a sizeable jump in shear strain across a line of one element width, extending a substantial distance from the crack tip at an angle of 90° from the crack line. Yet the analysis of Case 1 in Section 5.1 above explicitly and clearly rules out the possibility of a strain jump under such conditions!

VS were aware of this disagreement of their solution with the results obtained independently by Drugan and Shen (1987) and Leighton *et al.* (1987), and they showed that the stress and deformation paths through the "shock" in their solutions do not satisfy one or both of (45) and (49) above. This is true both for their numerical crack growth solutions, and for the discontinuity example they propose in their Appendix A. Drugan and Shen (1987) and Leighton *et al.* (1987) obtained (45) and (49) by heuristic means: they argued that since the jump forms of those equations must hold across a shock, i.e.,

$$[\![\mathbf{n}\cdot\boldsymbol{\sigma}]\!] = -\rho c[\![\mathbf{v}]\!], \tag{82}$$

$$\llbracket \varepsilon \rrbracket = -\frac{1}{2c}(\llbracket \mathbf{v} \rrbracket \mathbf{n} + \mathbf{n} \llbracket \mathbf{v} \rrbracket), \tag{83}$$

these imply that (45) and (49) should constrain the stress, strain and velocity paths *through* the shock. Thus, VS's argument that (82) and (83) do not *demand* (45) and (49) was perhaps plausible at the time their paper was written.

Now, however, the situation is different. Section 3.2 above proves that a shock wave propagating under steady-state, nonrotating conditions is identical to a suitably-chosen steady-state smooth wave in arbitrary materials. Then, Section 4.1 proves that momentum conservation and compatibility rigorously require that the stress and deformation paths satisfy (45) and (49) through a smooth wave which propagates under steady-state, nonrotating conditions, when the variations of field variables along the wave are negligible compared to their variations through the wave (since the wave is to model a shock, where this is obviously true). These results are also, obviously, valid for arbitrary materials. Thus, there no longer appears to be a plausible way to argue that (45) and (49) need not be satisfied through VS's proposed "shocks", since their numerical solution imposes steady-state conditions, and the "shocks" observed do indeed propagate without rotating.

An additional argument in support of this conclusion is the following: Although the elastic–plastic material models treated in VS are rate-independent, and hence if they could sustain shocks such would be mathematically sharp, any real material for which the VS solutions are believed applicable will actually possess some (possible minute) rate-*dependence*, i.e., some viscosity. A shock in such material will have finite thickness—that is, it will actually be a propagating narrow transition zone, through which continuum field quantities vary rapidly but continuously. Thus [and this has always been the viewpoint of researchers in the shock wave community, as discussed e.g. by Courant and Friedrichs (1948), Whitham (1974) and Smoller (1983)], any shock in an inviscid material model should differ negligibly from one in a material model that is identical except that it possesses infinitesimal viscosity. Again, a shock in this latter model will be a narrow transition zone through which field quantities vary rapidly but continuously. Within such a transition zone, *regardless of the material model*, we have proved that conditions (45) and (49) would *have* to be satisfied when the zone propagates under steady-state conditions. Thus, if one were to re-analyze numerically exactly the same problem treated by VS, except that their constitutive model now contained a minute amount of rate dependence, what appears as a shock in their 1994 paper would necessarily become a narrow (finite-width) zone of rapid, continuous transition which propagates under steady-state conditions without rotating. Our Section 4.1 analysis applies directly to this, proving that (45) and (49) *must* be satisfied through such a continuous transition zone. Thus, the intra-shock stress/strain/velocity paths actually calculated in the rate-dependent numerical solutions (presuming of course accurate numerical solution of the fields through the narrow transition zone) would differ substantially from those implied by the VS solutions, which VS themselves point out significantly violate (45) and/or (49). Thus, the rate-independent results of VS would necessarily differ significantly from the results with minute rate-dependence, which is physically unacceptable.

One might be tempted to argue that one assumption of the present analysis, namely that variations of field variables along the wave are negligible compared to their variations through the wave, is violated near the growing crack tip, which VS's calculations imply is the site of significant strain gradients. Although this might necessitate further contemplation very close to the crack tip, the "shocks" exhibited by VS's solutions extend, e.g. in the nonhardening case illustrated in their Figure 1, out to a radius on the order of 1/3 the maximum plastic zone dimension. Clearly ours is an excellent assumption over most of this "shock's" expanse, as VS always found its width to be one element thick, regardless of mesh refinement.

The inescapable conclusion appears to be that the discontinuity surfaces present in the VS computations cannot correspond to actual shock waves in a real material. This leads to two possibilities: one is that there do exist *fully continuous* solutions to the steady-state dynamic crack growth problems studied by VS which their numerical method was unable to obtain [for example, the full-field counterpart of the shock-free steady-state asymptotic solution derived by Leighton *et al.* (1987)]. The other possibility is more tantalizing: that *steady-state* dynamic plane strain crack growth is not possible, at least in the material model analyzed here. The resolution of this issue should perhaps be attempted by a different full-field numerical method which is capable of solving both the nonsteady as well as steady-state dynamic crack propagation problems, and which for steady-state growth incorporates the thermodynamics-mandated shock restrictions derived here.

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