U. Wisconsin, R. Lakes

Scaling in bending

The ratio of stiffness to density E/ρ of a material is important in many applications, however for bending, it is of interest to examine E/ρ^2 or E/ρ^3. For example we consider the deflection δ of a rod of length L, Young's modulus E, and circular section of radius r, and under a load per unit length w.

$$\delta = wL^4/8EI, \quad (1.2)$$

with $I = \pi r^4/4$. (1.3)

Since the mass is m $\rho\pi r^2 L$, the rigidity per unit mass is

$$\Gamma = \frac{w/\delta}{m} = \frac{8EI}{L^4 m} = \frac{2Er^2}{L^5\rho} \quad (1.4)$$

Let the rod radius r be free to vary; eliminate it via the equation for density. Then the rigidity per unit mass is

$$\Gamma = \frac{2m}{L^6\pi}\ \frac{E}{\rho^2} \quad (1.5)$$

So, for a rod in bending, the rigidity per unit mass goes as the ratio of stiffness to the square of the density. Similarly, for a plate in bending the rigidity goes as the ratio of stiffness to the cube of the density. For this reason cellular solids can be much more rigid per unit mass in bending than fully dense materials, even high performance ones, as shown in Table 1.

Such concepts of scaling are well known in homogeneous materials and in cellular solids of 'conventional' or non-hierarchical structure (Gordon, 1983, Schmidt-Nielsen, 1993). However the effects of scaling in engineering type hierarchical solids are not yet well understood and remain to be explored.

Table 1 Stiffness, density, and performance ratios for several materials

Wood has the best performace ratio in rod bending and in plate bending. Bone, however, has self repair capability.

Material	E[GPa]	ρ(Mg/m^3)	E/ρ	E/ρ^2	E/ρ^3
steel	210	7.8	26.9	3.45	0.44
titanium	120	4.5	26.7	5.93	1.32
aluminum	73	2.8	26.1	9.31	3.33
magnesium	42	1.7	24.7	14.5	8.55
wood (spruce)	14	0.5	28.0	56	**112**
Bone	14	2	7	3.5	1.75
graphite-epoxy cross ply	83	1.54	53.9	35	23