# Extreme Cosserat elastic hinged lattices with Sarrus links

R. S. Lakes*
* Department of Engineering Physics and Department of Materials Science,
University of Wisconsin, Madison, WI 53706

* Corresponding author. rlakes@wisc.edu

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**Abstract**

Hinged lattices are envisaged with Sarrus linkages in the ribs and with rigid nodes where the ribs intersect. Triangulated lattices with Sarrus hinged ribs only deform volumetrically and have a Poisson's ratio of -1. Lattices containing squares and hexagons can deform axially as well as volumetrically. All the lattices are compliant with respect to tension and compression and rigid with respect to bending. The lattices exhibit extreme Cosserat elasticity corresponding to rotational elasticity.

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## 1 Introduction and rationale

Hinged lattices have been envisaged in the context of extremal elastic behavior such as a negative Poisson's ratio approaching the lower isotropic limit -1. For example, a structure containing ideal sliders and hinges was predicted to exhibit a Poisson's ratio of -1 [1]. Flexible foams were made with a modified cell shape; they had a Poisson's ratio of -0.7 or lower [2]; other elastic materials were subsequently found or designed to have a negative Poisson's ratio. Arrays of rotating hexamers were studied in the context of negative Poisson's ratio [3, 4]. Two dimensional hexagonal complex structures with hinge joints or sliders [5] exhibiting Poisson's ratio close to -1 were envisaged and analyzed. The aim was to develop two phase composites with Poisson's ratio tending to -1. Such composites were hierarchical, with structure on very different length scales. Three dimensional variants were also considered. Numerical homogenization with the aim of obtaining extremal Poisson's ratios resulted in several 2D hinged lattice structures with Poisson's ratio +1 and -1 [6]. Two dimensional lattices of hinged rotating squares were studied [7] in the context of negative Poisson's ratio in crystal physics. A two dimensional lattice of hinged rigid rotating squares [8] [9] was studied; these lattices were predicted to exhibit a Poisson's ratio of -1. This is the lower isotropic limit in two dimensions and in three dimensions [10]. Hinged lattices containing triangles [11], rhombi [12], blocks [13] and cubes [14] have also been determined to exhibit a negative Poisson's ratio. In the various analyses of hinged lattice structures, it is assumed that the hinges are ideal: that they have no friction and no resistance to rotation. Moreover, it is assumed that the lattice elements are perfectly rigid. These assumptions allow considerable simplification of analysis in comparison with lattices with flexible rib or plate elements.

Lattices comprised of ribs or plates are discontinuous by their nature but they are routinely treated using elasticity theory which is a continuum theory. Lattices of atoms are not continuous either. A continuum view is considered to be warranted if the structural elements are sufficiently small compared with the size of the model to be analyzed or the size of the object to be studied in the laboratory. The theory of elasticity is not the only continuum theory. Cosserat elasticity, among other theories, has more freedom than the usual theory of elasticity which is called classical. Cosserat solids [15] [16] have a local rotation variable and are sensitive to gradients of rotation; they incorporate moment per area or couple stress as well as force per area. Behavior in torsion or bending will differ from predictions based on classical analysis using moduli obtained in the absence of gradients. Other generalized continuum theories include the theory of elastic materials with voids [17] in which the solid is sensitive to gradients in dilatation and the Mindlin microstructure theory [18] allows the local rotation and strain to differ from the macroscopic rotation and strain. This microstructure theory provides more freedom than classical or Cosserat elasticity. In the present research, lattices containing ribs with hinged Sarrus elements are studied. They are found to be strongly nonclassical. In particular they are found to be stiff in bending compared with tension corresponding to strong Cosserat elasticity.

# 2 Lattices with Sarrus hinged linkages

## 2.1 Rib elements

Lattices are structures with cells that repeat in a spatial array. Lattices based on a repetitive structure of atoms represent crystals. On a larger scale, lattices may be envisaged with slender rib elements; these are called rib lattices or truss lattices. The ribs may stretch, bend, or twist as the lattice is deformed. Rib lattices are of interest in the context of the freedom they offer in connection with isotropic or anisotropic response, and in applications in lightweight materials and structures. Depending on the structure, lattice deformation may be dominated by stretching or bending of the rib elements.

A rib element with a Sarrus linkage [19] [20] is shown in (Figure 1). The rib cross section can be square as shown or it can be triangular. The hinges in this linkage allow axial extension or squeezing but do not allow bending or twisting.

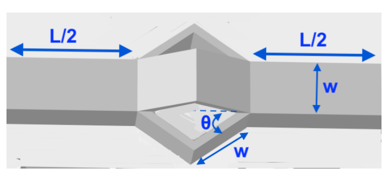


Figure 1: Idealized lattice rib with a hinged Sarrus linkage.

Assume that the hinges are frictionless, and that the material comprising the rib is rigid (its elastic modulus tends to infinity). Also assume that any nodes connecting ribs are also rigid so that no relative rotation between ribs is allowed. If the hinged structure evokes the notion of a structural mechanism rather than a material, then one may also assume the rib length is smaller than the near point resolution of the eye, about 0.1 mm, so one does not see the structure.

The Sarrus linkage is not unique in allowing stretching without allowing bending. One may envisage nested tubes that can slide with respect to each other without friction. Round tubes were presented in the context of extreme couple stress materials [21]. Round nested tubes will resist bending but allow extension and torsion. If rigidity with respect to torsion is desired, the tube cross sections can be made noncircular. Tube based ribs will come apart under tension. This could be prevented by separating the tubes with a layer of ball bearings and providing annular flanges to limit the extension.

The hinged plates in the linkage are assumed to be at angle $\theta$ with respect to the rib axis. The straight rib segments have length $L/2$; the rib width is $w$. Referring to Figure 1, the hinged section has length $L_s = 2wcos\theta$ so the total length of the rib is $L_{tot} = L + 2wcos\theta$. The rib cross section can be square as shown or it can be triangular. The overall axial strain of the rib is, with $\theta_0$ as an initial angle is

$$\epsilon = \frac{2w(cos\theta - cos\theta_0)}{L + 2wcos\theta_0}. \tag{1}$$

If one seeks to maximize allowable strain, $L$ can be reduced in comparison with $w$. Such reduction is limited by the requirement that the Sarrus elements fit in the lattice without contact.

### 2.2 Triangulated lattices

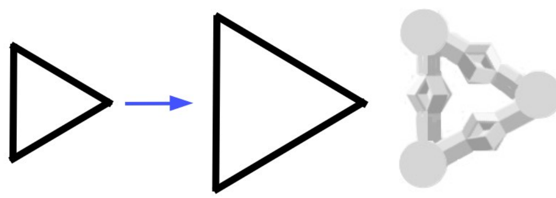

Figure 2: Triangle with ribs that can change length but not angle: only change in area occurs, not change in shape. Right: triangle with Sarrus ribs.

Envisage a triangle of such ribs. Any deformation of this triangle can give rise to a change of area alone with no change of shape (Figure 2). A shape change would require a change in angles between ribs which is excluded by the assumption of node rigidity.

Now envisage a fully triangulated structure that contains only triangular elements of ribs with Sarrus hinged linkages. Because each element can only change in area, a 2D structure of only triangular elements can also only change in area, not shape. The Poisson's ratio is -1. Similarly a fully triangulated 3D rib lattice structure can only change in volume with no change in shape, so Poisson's ratio is also -1. Each rib undergoes the same axial strain so the overall lattice undergoes the same strain. If ribs of different length are incorporated in a lattice, the initial length occupied by the Sarrus segment is assumed to be the same fraction of the overall rib length for all ribs (equation 1). Examples of fully triangulated 3D rib lattice structures include the octet truss, the body centered cubic lattice and the face centered cubic lattice. An illustration of a face centered cubic lattice with Sarrus ribs is shown in Figure 3.

There is only one easy mode of deformation, a volume change. These lattices will be easy to stretch or compress with a Young's modulus of zero in any direction. They will also be easy to deform volumetrically with a bulk modulus of zero.

This behavior of hinged triangulated rib lattices is very different from that of flexible triangulated rib lattices with uniform elastic ribs. For example the octet truss with slender elastic ribs is anisotropic [22] with Poisson's ratios in different directions from 1/6 to 1/3; elastic moduli are nonzero in all directions. In the analysis of this elastic rib lattice the joints are assumed to be pin joints rather than rigid nodes so relative rotation of the ribs occurs; the angle between ribs at the

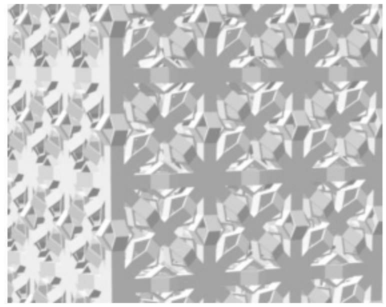

Figure 3: Face centered cubic lattice with Sarrus ribs.

joints is not constant. Similarly a face centered cubic rib lattice exhibits nonzero elastic moduli in all directions [23].

### 2.3 Lattices with squares and hexagons

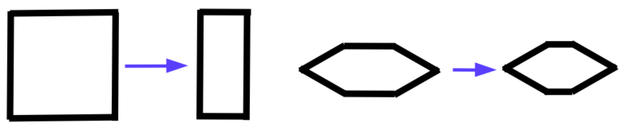

Figure 4: Squares and hexagons with ribs that can compress. Angles between ribs do not change. These figures can change in shape and undergo axial deformation as shown or they can change in area.

Now envisage lattices with the same ideal Sarrus links as above but containing arrays of ribs as squares, hexagons or other shapes. As above, the nodes are rigid so the angle between ribs does not change. In such cases, the area can change but the shape can change as well as illustrated in the following. A square can be stretched in a principal direction by stretching only the ribs in that direction with no effect on the orthogonal ribs. Similarly, a hexagon can be deformed by stretching two parallel ribs with no effect on the other four ribs. These deformations are illustrated in Figure 4.

Two dimensional lattices comprised of such squares or hexagons will also have the freedom to deform in one direction but need not deform in the orthogonal direction. In contrast to triangulated lattices they need not be restricted to a change only in area.

Three dimensional lattices containing squares or hexagons can have similar freedom. For example, the truncated octahedron or tetrakaidecahedron can be packed to form a lattice. This polyhedron, if provided with ideal Sarrus link ribs, can deform volumetrically by equal extension of all ribs but it can also deform axially as shown in Figure 5. Squares and hexagons in the lattice can deform in one direction without changing angles between ribs as indicated by the paired arrows. So the lattice cells can change shape while preserving angles between ribs. Similarly, in a lattice with diamond structure in which ribs can compress but angles do not change, the lattice can deform in one direction without changing angles between ribs as shown in Figure 6. It can also change in volume if all the ribs extend or compress.

Figure 7 shows a three dimensional hexagonal lattice in which the hexagons in transverse planes are triangulated. The Poisson's ratio for deformation in the transverse plane shown in the figure

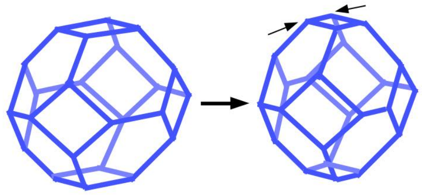

Figure 5: Truncated octahedron lattice cell. Ribs can compress but angles between ribs do not change. Axial deformation is possible as well as axial deformation.

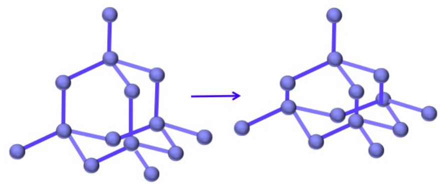

Figure 6: Diamond structure lattice. Ribs can compress but angles between ribs do not change. The lattice can undergo axial deformation as well as volumetric deformation.

will be -1. Volumetric deformation of this lattice is an easy mode. Young's modulus for deformation in any direction is zero. The orthogonal ribs that link the hexagonal layers shown in the oblique

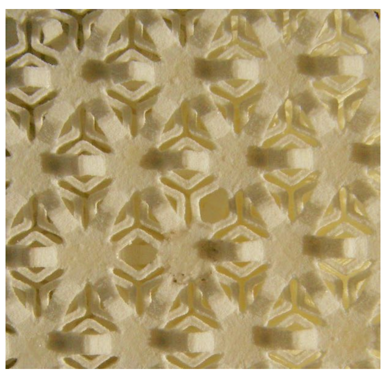

Figure 7: Hexagonal triangulated lattice with Sarrus ribs, transverse plane.

view in Figure 8 can deform independently, so the lattice is anisotropic. The Poisson's ratio for load in the axial direction will be near zero.

## 2.4 Bending and torsion

In the triangulated lattices, there can clearly be no shear hence no torsion because the angle in each triangle is fixed. In the other lattices, one can have extension in one direction and compression in orthogonal directions, so shear can occur in non-principal directions. The lattices are nevertheless rigid with respect to torsion because the rib elements cannot twist. The ribs cannot pivot at the nodes either, because they are assumed to be rigid.

In all these lattices, because the Sarrus links do not admit bending or torsion and the nodes

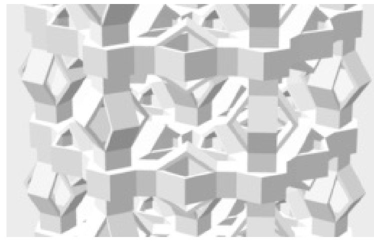

Figure 8: Hexagonal triangulated lattice with Sarrus ribs, oblique view.

are rigid with respect to rotation, the lattices will be rigid with respect to bending despite the zero Young's modulus in tension. Bending, even in an anisotropic elastic solid, is governed only by the Young's modulus in the bar direction. Such a disparity between bending and tension response is not consistent with the theory of elasticity. It is, however consistent with Cosserat elasticity in the extreme limit of diverging characteristic length. Similarly, a disparity between response in homogeneous shear and torsion is inconsistent with the classical theory of elasticity but it is consistent with Cosserat elasticity.

# 3 Cosserat elasticity

Cosserat elasticity [15] predicts different response in bending vs. tension because, in contrast to classical elasticity, there is sensitivity to strain gradient. Cosserat solids with a local inertia variable are called micropolar [16]. The constitutive equations for linear isotropic Cosserat elasticity are:

$$\sigma_{ij} = 2G\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} + \kappa e_{ijk}(r_k - \phi_k) \tag{2}$$

$$m_{ij} = \alpha\phi_{k,k}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i}. \tag{3}$$

The points in a Cosserat solid can undergo a micro-rotation $\phi_k$. This local rotation may differ from $r_k = \frac{1}{2}e_{klm}u_{m,l}$ called the "macro" rotation obtained from the antisymmetric part of gradient of displacement $u_i$. $e_{jkm}$ is the permutation symbol. The stress $\sigma_{ij}$ (force per unit area) in Cosserat elasticity can be asymmetric. By contrast, in classical elasticity, the stress is symmetric. Cosserat theory incorporates a couple stress $m_{ij}$ (a torque per unit area) which balances the distributed moment from the asymmetric stress.

Isotropic Cosserat solids have six elastic constants. These are $\lambda$, $G$, $\alpha$, $\beta$, $\gamma$, $\kappa$. Constants $\lambda$ and $G$ have the same meaning as in classical elasticity. $G$ is the shear modulus in the absence of gradients. Constants $\alpha$, $\beta$, $\gamma$ allow sensitivity to rotation gradient. Couple stress theory corresponds to $\kappa \to \infty$.

An exact solution is known for bending of a plate to a cylindrical shape [25]. The rigidity ratio $\Omega = \frac{MR}{EI}$ is the ratio of the Cosserat bend rigidity to the classical value. $M$ is the applied moment, $R$ is the radius of curvature, and $E$ is Young's modulus. The rigidity ratio with $w$ as the bar full width and depth, is, with $\nu$ as Poisson's ratio,

$$\Omega = 1 + 24(\frac{\ell_b}{w})^2(1-\nu). \tag{4}$$

Size effects occur in bending. Rigidity in bending exceeds the value expected from classical analysis using the Young's modulus obtained from tension or compression in the absence of gradients. The size effect depends on the Cosserat characteristic length for bending $\ell_b = \sqrt{\frac{\gamma}{4G}}$. Bending analyses are also available for Cosserat elastic round rods and square cross section bars. These have been used in interpreting experiments on bone, foams and lattices. Similar size effects are predicted in torsion [25]; they are governed by the torsion characteristic length $\ell_t = \sqrt{\frac{\beta+\gamma}{2G}}$. Rigidity in torsion exceeds the rigidity predicted with classical elasticity from the shear modulus obtained in the absence of gradients. The larger the characteristic length, the larger the difference between predictions of Cosserat and classical elasticity.

As for other generalized continuum theories, the theory of elastic materials with voids [17] predicts size effects in bending but not in torsion so it cannot represent these lattices. It is also insufficient to represent any of the nonclassical materials studied in our laboratory. All of them exhibit size effects in both torsion and bending; they were interpreted with Cosserat elasticity. Microstructure elasticity theory [18] incorporates more freedom than Cosserat elasticity; there are 18 isotropic elastic constants rather than 6. Such freedom cannot be excluded either in the present lattices or in prior studies of Cosserat solids. However an experimental paradigm to reveal the 18 constants from experiment is not available. Also, a hexagonal lattice with pin jointed ribs (different from the present lattices) has been homogenized analytically to a second gradient elastic theory [26]. Such theories could be experimentally compared with Cosserat elasticity only if solutions of suitable deformation problems are provided.

The hinged Sarrus lattices are rigid with respect to bending and torsion but are compliant with respect to some homogeneous deformations. They therefore behave as extreme Cosserat solids with characteristic lengths that tend to infinity.

### 3.1 Physical lattices

Idealized hinged lattices with many cells cannot be realized in the physical world. However a macroscopic hinge can be made with extremely low friction by using ball bearings, roller bearings or air bearings. Such an approach would be challenging to embody for a lattice with the hundreds or thousands of hinges required to justify its study as a continuum. In any case, ribs made of a physical material will not be perfectly rigid.

One can, however, make flexible links that in some respects resemble hinges. For example, lattices were designed with ribs containing flexible substructures inspired by the Sarrus linkage. These lattice were fabricated via 3D printing and studied experimentally [27] [28] for classical and nonclassical elastic response. The lattices exhibited Cosserat elastic response. A face centered cubic 3D printed polymer lattice [28] based on the structure in Figure 3 exhibited Cosserat size effects between a factor of two and about a factor of 4; characteristic lengths were less than the cell size. A 3D printed polymer lattice with triangulated hexagonal structure based on the ideal structure in Figure 8 exhibited strong nonclassical Cosserat size effects [27] of a factor of 30 in bending and torsion compared with classical predictions. Specifically, effective moduli in bending of slender specimens were about a factor 30 greater than anticipated based on moduli measured in compression. Nonclassical effects of even larger magnitude can be achieved following advances in 3D printing. In the limit, lattices with similar structure but with ideal hinges would exhibit nonclassical effects of magnitude tending to infinity.

It may also be possible to approximate the nested tube concept in flexible materials as follows. Nested tubes of different diameter could be attached by thin annular discs, compliant in tension and compression but relatively stiff in torsion. If there are several such discs the tube system will be stiff in bending as well.

## 4 Discussion

Hinged lattices with Sarrus links deform easily in area or volume. Some of these lattices can also deform easily for tension or compression in one direction. All the lattices considered are rigid with respect to bending. They do not obey the classical theory of elasticity. Sensitivity to deformation gradient can be understood in the context of Cosserat elasticity. The lattice cells can be made small but not infinitesimal. If the cells are smaller than the resolution limit of the human eye, the solid appears to be a homogeneous material. Even so, one cannot attain the continuum limit but physical materials are not continua either because atoms have a nonzero size. If the lattices are ideal, that does not matter because the characteristic lengths still diverge.

Lattices with resistance to rotation are not entirely new; they were envisaged in 1890 by Kelvin [24] who considered a lattice of pivoted ribs provided with gyroscopes to incorporate resistance to rotation. The motivation was to create a mechanical model for the aether to visualize propagation of electromagnetic waves. As with ideal hinged lattices, these lattices were primarily conceptual rather than a direct design for physical embodiment. Nevertheless, such conceptual models can provide inspiration for development of physically realizable heterogeneous materials. Indeed, lattices with rigid nodes where ribs intersect correspond more closely with realizable elastic lattices than idealizations with pin joints connecting the ribs.

The strain in lattices containing Sarrus ribs is limited by the fraction of the rib that is occupied by the Sarrus linkage as indicated in Equation 1. Strain is also limited by geometrical constraints in lattices with rotating squares [8] or other polygons. It is possible to obtain large strain in other ideal hinged lattices by use of geometry that multiplies motion [29].

## 5 Conclusion

Hinged lattices that contain ribs with ideal Sarrus links can have a Poisson's ratio of -1 if the structure is triangulated. All lattices with Sarrus links studied here stretch freely but are rigid with respect to bending and torsion. They may regarded as extremal Cosserat solids.

## 6 Acknowledgment

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