Rayleigh-Taylor Instability Notes

Jason Oakley
January, 2004

Audience: Engineering students, with some fluid mechanics background, who are interested in the instability with an introduction to perturbation analysis of the fluid dynamics equations of motion.

References:
1. Chandrasekhar, S., Hydrodynamic and Hydromagnetic stability, Dover, 1981, first published by Oxford University Press, 1961.
2. Rayleigh, "100. Investigation of the character of the equilibrium of an incompressible heavy fluid of variable density," Scientific Papers by Lord Rayleigh, Dover, 1964.
3. Taylor, G, "The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I," Proceedings of the Royal Society, A, vol. CCI, 1950, p 192-196.

Introduction

"… two fluids of different densities superposed one over the other (or accelerated towards each other); the instability of the plane interface between the two fluids, when it occurs, is called the Rayleigh-Taylor instability" [1].

"If the horizontal surface of a liquid at rest under gravity is displaced into the form of regular small corrugations and then released, standing oscillatory waves are produced. Theoretically, a liquid could exist in a state of unstable equilibrium with a flat lower horizontal surface supported by air pressure" [3].

The physics involved are rather intuitive: "heavy stuff falls, light stuff rises". The intuitive nature of stability can be seen with a ball at the bottom of a valley contrasted to the unstable situation of a ball set on top of a hill. A ball on the top of a hill, given a slight nudge in any direction, will continue in that direction gaining speed until it reaches the bottom of the hill. A ball at the bottom of valley will return to its initial position after a small displacement; however, the ball may very well oscillate about its initial position until the dissipative force of friction brings it to rest. An analogous situation would be a spherical balloon filled with a gas heavier than air, e.g., carbon dioxide, replacing the ball. An inverse situation may be a spherical balloon filled with helium in a stable situation at the ceiling of a cave, or in an unstable situation, balancing precipitously on the tip of a stalactite where any displacement will cause the balloon to rise to the ceiling.

In the context of the Rayleigh-Taylor instability, there are two fluids being considered, one above the other, which are separated by a horizontal plane. If the light fluid is above the heavy fluid, the interface between the two is "stable" because the heavy stuff may be considered to have already fallen and the light stuff has already risen. Stable does not imply there is no motion, if there is air above a water surface, a displacement of the surface will result in waves traveling horizontally on the surface. For a heavy fluid over a light fluid, the heavy stuff wants to fall and the light stuff wants to rise; however, for a perfectly flat interface there are no avenues for this vertical motion to occur. When this perfectly flat interface is displaced (perturbed) the waves generated (as seen in the stable case) provide the necessary means for the light fluid to rise (in the peak of the wave) and the heavy fluid to fall, in the valley of the wave. These notes will quantify this motion.

To begin we will consider a large box filled with a fluid with the coordinate system shown as is Fig. 1.

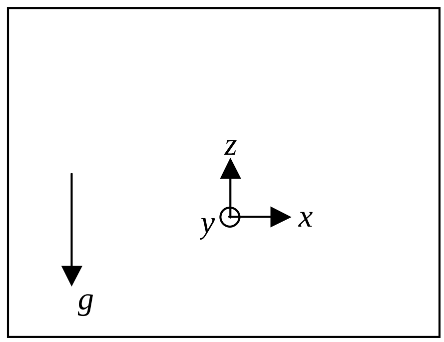

Figure 1. Fluid inside of a large box.

The coordinate system has *x* as the horizontal, *z* as the vertical, and *y* as going into the page. The acceleration of gravity, *g*, is in the negative vertical direction. The Navier-Stokes equations govern the motion in fluid dynamics and are a set of five coupled, conservation equations: mass, momentum (one equation for each of the three principal directions), and energy.

Conservation of Mass

The continuity, or conservation of mass, equation on a per volume basis is:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \left(\rho \underline{V} \right) = 0 \,,$$

where ρ is density, *t* is time, $\underline{\nabla}$ is the vector differential operator, and $\underline{V}$ is the velocity vector. The continuity equation is expanded as:

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(\rho U \hat{i} + \rho V \hat{j} + \rho W \hat{k} \right) = 0 \,,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho U \right) + \frac{\partial}{\partial y}\left(\rho V \right) + \frac{\partial}{\partial z}\left(\rho W \right) = 0 \,,$$

where *U*, *V*, and *W* are the velocity components in the *x*, *y*, and *z* directions. Now we will make our first assumption: the system we are interested in is 2-dimensional. Thus, we will neglect any changes in the *y*-direction and this makes any derivatives with respect to *y* equal to zero (the third term in the above equation). The above equation reduces to:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho U)+\frac{\partial}{\partial z}(\rho W)=0 .$$

Now expand this equation:

$$\frac{\partial \rho}{\partial t}+U\frac{\partial \rho}{\partial x}+\rho\frac{\partial U}{\partial x}+W\frac{\partial \rho}{\partial z}+\rho\frac{\partial W}{\partial z}=0.$$

Our next assumption will be one of incompressibility for a given layer of fluid, that is, a horizontal plane of fluid at a given z coordinate will have a constant density. Thus, changes in density (with respect to time and space) are zero meaning the temporal and spatial derivatives are zero. The above equation reduces to:

$$\rho\frac{\partial U}{\partial x}+\rho\frac{\partial W}{\partial z}=0.$$

Now, dividing through by density we have:

$$\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}=0.$$

A couple of notes about this equation are: 1) we have employed the "Boussinesq approximation" which is another way of saying the density is assumed constant, and 2) the equation is said to be "solenoidal" as the definition of a solenoidal vector is one whose divergence is zero, i.e., $\underline{\nabla}\cdot\underline{V}=0$.

The continuity equation is almost in its final form for the analysis, and the final step is to write the velocity in terms of a perturbed quantity. We will consider that the fluid is initially at a constant state and only consider small changes from this state:

$$U=U_0+u$$
$$V=V_0+v$$
$$W=W_0+w,$$

where the subscript "0" indicates the constant initial value and the lower-case letters are small velocity deviations. The deviations in velocity are noted to be functions of both time and space, e.g., $u=u(x,y,z,t)$. Substituting into the above solenoidal equation:

$$\frac{\partial(U_0+u)}{\partial x}+\frac{\partial(W_0+w)}{\partial z}=0,$$
$$\frac{\partial U_0}{\partial x}+\frac{\partial u}{\partial x}+\frac{\partial W_0}{\partial z}+\frac{\partial w}{\partial z}=0,$$

and noting that derivatives of constant values are zero, we have the final form of the continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0.$$

Conservation of Momentum

The derivative operator *D*/*Dt*, sometimes called the material, substantial, or total derivative is defined as:

$$\frac{Dq}{Dt}=\frac{\partial q}{\partial t}+U\frac{\partial q}{\partial x}+V\frac{\partial q}{\partial y}+W\frac{\partial q}{\partial z},$$

where q may be a scalar, such as density, or a vector, such as velocity. We employ the total derivative when using the Eulerian point of view (as opposed to the Lagrangian). From this perspective we analyze a given point in space, perhaps an infinitesimally small cube of dimensions $dx \times dy \times dz$, and the first term is how much q is changing in time and the last three terms are how much q is being convected through this small volume. If q is the velocity, then we have the Eulerian acceleration.

Newton's second law of motion (force equals mass times acceleration) on a per volume basis is:

$$\rho \frac{D\underline{V}}{Dt} = F_{BODY} + F_{SURFACE},$$

where the body force we will consider is due to gravity, and the surface forces are due to pressure and shear. The momentum equation then becomes:

$$\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\tau}},$$

where the acceleration vector is $\underline{g} = 0\hat{i} + 0\hat{j} - g\hat{k}$, p is the pressure, and $\underline{\underline{\tau}}$ is the viscous stress tensor. We will say that $\underline{\underline{\tau}} = \mu(\cdots)$ and employ the inviscid assumption (μ=0) so we will neglect the viscous stress tensor term. The momentum equation becomes:

$$\rho \frac{D\underline{V}}{Dt} = \rho \underline{g} - \underline{\nabla} p.$$

The x-momentum equation is the first of the three momentum equations:

$$\rho \frac{DU}{Dt} = \rho g_x - \frac{\partial p}{\partial x},$$

and noting that g_x=0 and expanding the total derivative:

$$\rho \left(\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} \right) = -\frac{\partial p}{\partial x}.$$

The pressure and density will be rewritten in terms of average values (ρ and p) with an additional, small, "perturbed" quantity, $\delta\rho$ and δp. The average values or ρ and p are constant within a given fluid layer at a constant z, however, due to the hydrostatic pressure gradient, they are both functions of height. The perturbed density $\delta\rho$ is considered incompressible so that as a small element of increased density fluid moves through the fluid, this small element's density will not change. Therefore, $\rho=\rho(z)$ and $p=p(z)$ and $\delta\rho=\delta\rho(x,y,z,t)$ and $\delta p=\delta p(x,y,z,t)$. Substituting these into the x-momentum equation along with the decomposed velocities gives:

$$(\rho + \delta\rho)\left(\frac{\partial (U_0 + u)}{\partial t} + (U_0 + u)\frac{\partial (U_0 + u)}{\partial x} + (V_0 + v)\frac{\partial (U_0 + u)}{\partial y} + (W_0 + w)\frac{\partial (U_0 + u)}{\partial z} \right) = -\frac{\partial (p + \delta p)}{\partial x},$$

and expanding the velocity terms and neglecting derivatives with respect to y:

$$(\rho + \delta\rho)\left(\frac{\partial U_0}{\partial t} + \frac{\partial u}{\partial t} + U_0 \frac{\partial U_0}{\partial x} + U_0 \frac{\partial u}{\partial x} + u \frac{\partial U_0}{\partial x} + u \frac{\partial u}{\partial x} + W_0 \frac{\partial U_0}{\partial z} + W_0 \frac{\partial u}{\partial z} + w \frac{\partial U_0}{\partial z} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial (p + \delta p)}{\partial x}.$$

The uppercase, subscripted velocities have been assumed to be constant and their derivatives are set equal to zero in the above equation:

$$(\rho + \delta\rho)\left(\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} + W_0 \frac{\partial u}{\partial z} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial (p + \delta p)}{\partial x}.$$

At this point, another assumption will be made regarding the initial velocity condition. In our box of fluid we will assume that the initial velocity is zero for our Rayleigh-Taylor analysis, however, if we were conducting a Kelvin-Helmholtz (shear layer) instability analysis, this would be the correct *x*-momentum equation and the average velocity in the *x*-direction, U_0, would be non-zero. The x-momentum equation becomes:

$$(\rho+\delta\rho)\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial(p+\delta p)}{\partial x}.$$

Expanding the density and pressure terms, this equation becomes:

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\rho w\frac{\partial u}{\partial z}+\delta\rho\frac{\partial u}{\partial t}+\delta\rho u\frac{\partial u}{\partial x}+\delta\rho w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}-\frac{\partial \delta p}{\partial x}.$$

The next step is to linearize the equation. The justification is: changes in small quantities are initially themselves, small; therefore, a small quantity multiplied by a small quantity is much smaller than the initially small quantities and may therefore be neglected, e.g., $u(\partial u/\partial x)=0$. Finally, using this assumption and the fact that *p* is only a function of *z*, the x-momentum equation becomes:

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial \delta p}{\partial x}.$$

We will neglect the analysis of the y-momentum equation for this two-dimensional study. As an exercise, the student may start with the general form of the *y*-momentum equation and make a step by step analysis, as was done here for the *x*-momentum equation, and learn why this equation is inconsequential to our two-dimensional Rayleigh-Taylor study.

The *z*-momentum equation is:

$$\rho\frac{DW}{Dt}=\rho g_z-\frac{\partial p}{\partial z}.$$

Conducting a similar step by step analysis of this equation, and noting that our acceleration, $\underline{g}=-g\hat{k}$, the *z*-momentum equation reduces to:

$$\rho\frac{\partial w}{\partial t}=-(\rho+\delta\rho)g-\frac{\partial(p+\delta p)}{\partial z},$$

and expanding the density and pressure terms gives:

$$\rho\frac{\partial w}{\partial t}=-\rho g-\delta\rho g-\frac{\partial p}{\partial z}-\frac{\partial \delta p}{\partial z}.$$

The hydrostatic pressure, *p*(*z*), at any *z*-location is due to the weight of fluid above it and is related to the density by $\partial p/\partial z=-\rho g$. Using this in the above equation gives the final form of the *z*-momentum equation:

$$\rho\frac{\partial w}{\partial t}=-\delta\rho g-\frac{\partial \delta p}{\partial z}.$$

The last of the Navier-Stokes equations is the energy equation. We will neglect this equation by saying that there is no heat or work produced by, or transferred to, the system, there are no dissipative processes, and the system is isothermal.

A recap of the continuity, and x- and z-momentum equations:

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial \delta p}{\partial x},$$
$$\rho \frac{\partial w}{\partial t} = -\delta\rho g - \frac{\partial \delta p}{\partial z}.$$

We have three equations and four unknowns: *u, w,* δp, and $\delta\rho$. For our fourth equation, will once again employ the continuity equation but with the density perturbation, where previously we used the density, (and use the assumption that the initial velocity is zero):

$$\frac{\partial(\rho + \delta\rho)}{\partial t} + \frac{\partial}{\partial x}((\rho + \delta\rho)u) + \frac{\partial}{\partial y}((\rho + \delta\rho)v) + \frac{\partial}{\partial z}((\rho + \delta\rho)w) = 0,$$

Neglecting derivatives with respect to y and expanding:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \delta\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + \delta\rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + u\frac{\partial \delta\rho}{\partial x} + \rho\frac{\partial w}{\partial z} + \delta\rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} + w\frac{\partial \delta\rho}{\partial z} = 0,$$

and neglecting the nonlinear terms:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \delta\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial w}{\partial z} + w\frac{\partial \rho}{\partial z} = 0,$$
$$\frac{\partial \rho}{\partial t} + \frac{\partial \delta\rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + w\frac{\partial \rho}{\partial z} = 0.$$

Noting the term in parenthesis is zero from the first analysis of the continuity equation and that $\rho=\rho(z)$ (not *t* or *x*), the final form of our fourth equation is:

$$\frac{\partial \delta\rho}{\partial t} = -w\frac{\partial \rho}{\partial z}.$$

Review list of assumptions:

1. 2-dimensional, i.e., $\partial/\partial y=0$.
2. incompressibility for a given layer of fluid.
3. velocity rewritten in terms of initial constant values plus small deviations.
4. inviscid.
5. initial velocities are zero.
6. momentum equation is linearized.
7. neglect energy equation (isothermal, etc.)

Normal Mode Analysis

For our linear stability analysis we begin with an initially stationary flow and assume there are small perturbations to the velocity and the physical variables. The density and pressure are only functions of *z* and we consider a small disturbance in terms of a periodic wave along a plane at *z*=0. For a single mode disturbance (in our two-dimensional *x*-*z* plane) the amplitude (*A*) of a disturbance is described by:

$$A(x, z, t) = A_k(z, t)\exp(ikx),$$

where *k* is the wave number. [Note: recall the exponential relation of Euler's equation: $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$.] We will seek solutions in time for $A_k(z,t)$ in the exponential form:

$$A_k(z,t) = A_k(z)\exp(nt),$$

and the disturbance is then described as:

$$A(x,z,t) = A_k(z)\exp(nt)\exp(ikx),$$
$$A(x,z,t) = A_k(z)\exp(ikx + nt).$$

The requirement that the equations have a non-trivial solution to specified boundary conditions is a characteristic value problem (or eigenvalue problem) for the eigenvalue *n* corresponding to the wave number *k*. Instead of a single mode, we could have generalized this to a number of superposed modes which would then have independent solutions with a corresponding eigenvalue for each wave number. A positive eigenvalue will result in a positive exponential and an unstable situation and a negative eigenvalue will result in a stable situation- one that reduces the amplitude.

For our two velocities and density and pressure we have:

$$u = u(x,z,t) = u_k(z)\exp(ikx + nt),$$
$$w = w(x,z,t) = w_k(z)\exp(ikx + nt),$$
$$\delta\rho = \delta\rho(x,z,t) = \delta\rho_k(z)\exp(ikx + nt),$$
$$\delta p = \delta p(x,z,t) = \delta p_k(z)\exp(ikx + nt).$$

Recalling our four Navier-Stokes equations:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$
$$\rho\frac{\partial u}{\partial t} = -\frac{\partial \delta p}{\partial x},$$
$$\rho\frac{\partial w}{\partial t} = -\delta\rho g - \frac{\partial \delta p}{\partial z},$$
$$\frac{\partial \delta\rho}{\partial t} = -w\frac{\partial \rho}{\partial z},$$

and substituting in the four amplitude relations:

$$iku + \frac{\partial w}{\partial z} = 0, \tag{1}$$
$$\rho n u = -ik\delta p, \tag{2}$$
$$\rho n w = -\delta\rho g - \frac{\partial \delta p}{\partial z}, \tag{3}$$
$$n\delta\rho = -w\frac{\partial \rho}{\partial z}. \tag{4}$$

The only derivatives in the above equations are with respect to *z*, and therefore the partial notation is dropped. Multiplying Eq. (2) by *ik* and substituting (-*dw*/*dz*) for *iku* from Eq. (1) we have:

$$-\rho n\frac{dw}{dz} = k^2\delta p, \tag{5}$$

and solving for dp,

$$\delta p = -\frac{n}{k^2}\rho\frac{dw}{dz}, \tag{6}$$

and then substituting into Eq. (3):

$$\rho n w = -\delta\rho g + \frac{n}{k^2}\frac{d}{dz}\left(\rho\frac{dw}{dz}\right). \tag{7}$$

Solving Eq. (4) for dr:

$$\delta\rho = -w\frac{1}{n}\frac{d\rho}{dz}, \tag{8}$$

and substituting into Eq. (7):

$$\rho n w = w\frac{g}{n}\frac{d\rho}{dz} + \frac{n}{k^2}\frac{d}{dz}\left(\rho\frac{dw}{dz}\right), \tag{9}$$

and multiplying though by k^2/n:

$$\frac{d}{dz}\left(\rho\frac{dw}{dz}\right) - \rho k^2 w = -wg\frac{k^2}{n^2}\frac{d\rho}{dz}. \tag{10}$$

This is the governing differential equation we will use to solve for the instability between two fluids.

The Rayleigh-Taylor Instability for Two Incompressible Fluids

The box of fluid shown in Fig. 1 is now filled with two incompressible fluids of differing densities, separated by an interface with a perturbation imposed as shown in Fig. 2.

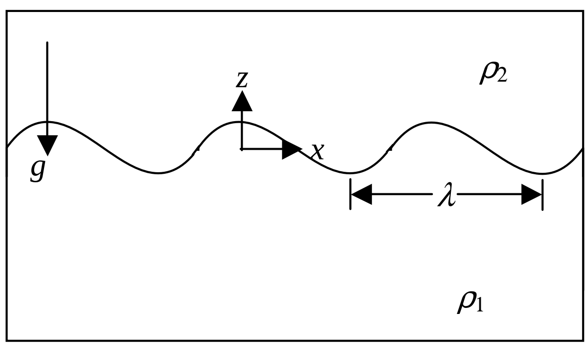

Figure 2. Two fluids inside of a large box.

For a layer of fluid above (ρ_2) or below (ρ_1) the interface, the density is constant and the governing differential equation, Eq. (10), reduces to:

$$\frac{d^2 w}{dz^2} - k^2 w = 0.$$

Using the boundary conditions that the velocity is zero at large distances above and below the interface, and the fact that the velocity at the interface is matched for the two solutions (kinematic constraint), we have:

$$w_2 = w_0 e^{-kz} \qquad (z > 0),$$
$$w_1 = w_0 e^{kz} \qquad (z < 0).$$

Next we apply Eq. (9) at the interface. First we multiply through by dz and integrate:

$$\int d\left(\rho\frac{dw}{dz}\right) - \int \rho k^2 w dz = -\int wg\frac{k^2}{n^2}d\rho.$$

Now integrating this across the interface, across an infinitesimal distance ($dz \approx 0$), the second term becomes zero, and the other two terms are:

$$\Delta\left(\rho \frac{dw}{dz}\right) = -wg\frac{k^2}{n^2}\Delta\rho,$$

$$\rho_2(-kw) - \rho_1(kw) = -wg\frac{k^2}{n^2}(\rho_2 - \rho_1),$$

$$-(\rho_2 + \rho_1) = -g\frac{k}{n^2}(\rho_2 - \rho_1),$$

and finally, solving for the eigenvalue n:

$$n = \sqrt{gk\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}}.$$

Thus, as our intuition initially told us, the system is unstable if the heavy fluid is above the lighter fluid ($\rho_2 > \rho_1$), because the eigenvalue is real, and stable if the light fluid is above the heavy fluid ($\rho_2 < \rho_1$), because the eigenvalue is imaginary. The quantity $(\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ is called the Atwood number (A), and a more compact form for the eigenvalue is then $n = \sqrt{gkA}$. For a positive Atwood number, the interface is unstable, and for a negative value it is stable. Finally, at x=0, at a peak of the perturbation, we find that the amplitude of a perturbation grows proportional to the quantity $\exp(t\sqrt{gkA})$. Thus, in the linear regime analyzed here, the growth rate is exponential in time, this should not be confused with the nonlinear growth rate, which is by some theories, linear with time.